EXPONENTS OF DIOPHANTINE APPROXIMATION IN DIMENSION TWO FOR A GENERAL CLASS OF NUMBERS

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ABSTRACT. We study the Diophantine properties of a new class of transcendental real numbers which contains, among others, Roy's extremal numbers, Bugeaud-Laurent Sturmian continued fractions, and more generally the class of Sturmian type numbers. We compute, for each real number ξ of this set, several exponents of Diophantine approximation to the pair (ξ, ξ^2), together with $\omega_2^*(\xi)$ and $\widehat{\omega}_2^*(\xi)$, the so-called ordinary and uniform exponent of approximation to ξ by algebraic numbers of degree ≤ 2. As an application, we get new information on the set of values taken by $\widehat{\omega}_2^*$ at transcendental numbers, and we give a partial answer to a question of Fischler about his exponent β_0.

1. INTRODUCTION

Given a real number ξ, we are interested in the following six classical exponents of Diophantine approximation: the exponent $\lambda_2(\xi)$ of simultaneous rational approximation to ξ and ξ^2, the dual exponent $\omega_2(\xi)$, the exponent $\omega_2^*(\xi)$ of approximation to ξ by algebraic numbers of degree at most 2, and the corresponding uniform exponents $\widehat{\lambda}_2(\xi)$, $\widehat{\omega}_2(\xi)$, $\widehat{\omega}_2^*(\xi)$ (the precise definitions are recalled in the next section, see also [10] and [11]). By [11, Theorem 2.3], for almost all real numbers ξ (with respect to the Lebesgue measure), we have

$$\lambda_2(\xi) = \widehat{\lambda}_2(\xi) = \frac{1}{2} \quad \text{and} \quad \omega_2(\xi) = \widehat{\omega}_2(\xi) = \omega_2^*(\xi) = \widehat{\omega}_2^*(\xi) = 2. \tag{1}$$

Moreover, if ξ is algebraic of degree at least 3, then (1) still holds as a consequence of Schmidt's subspace Theorem (see [11, Theorem 2.4]). There are currently few explicit families of transcendental numbers of which all exponents are known. Under the condition $\lambda_2(\xi) \leq 1$ (which excludes Liouville numbers, see [10, Corollary 5.4]), we have Roy's extremal numbers [28] and Fibonacci type numbers [30], Bugeaud and Laurent Sturmian continued fractions [11], and more generally the class of Sturmian type numbers [24] which generalizes the two last families. In some cases combinatorics on words provides numbers for which the six exponents can be computed. Given a word w written on the alphabet of positive integers we associate the real number $\xi_w = [0; w]$ whose partial quotients are successively 0 and the letters of w. Some combinatorial properties of w translate into Diophantine properties of ξ_w. For example, in [3] Allouche, Davison, Queffélec, and Zamboni proved that when w is a Sturmian or quasi-Sturmian sequence (see [22] and [13] for the definitions), then $\omega_2^*(\xi_w) > 2$, and thus ξ is transcendental. Bugeaud and Laurent [11] studied in depth the special case of Sturmian characteristic words. Their work generalizes a previous construction of Roy based on the Fibonacci word [27]. To state their result, let us recall some definitions. Fix an alphabet $\mathcal{A} := \{a, b\}$, where a, b are two distinct positive integers, and let $\mathcal{A}^*$ denote the monoid of finite words on $\mathcal{A}$ for the concatenation. Given an infinite sequence $\mathbf{s} = (s_k)_{k\geq 1}$ of positive integers or an irrational number $\varphi \in (0, 1)$ with continued fraction expansion $\varphi = [0; s_1, s_2, \dots]$, we define

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recursively a sequence of finite words $(w_k)_{k\geq 0}$ in $\mathcal{A}^*$ by

$$w_0 = b, \quad w_1 = b^{s_1-1}a \quad \text{and} \quad w_{k+1} = w_k^{s_{k+1}}w_{k-1} \quad (k \geq 1).$$

This sequence converges to an infinite word $w_\varphi = \lim_{k\to\infty} w_k$ called the *Sturmian characteristic word of slope φ on $\mathcal{A} = \{a,b\}$*. These words are important in combinatorics on words, see for examples [22], [17], [16]. We associate to w_φ the real numbers $\xi_\varphi := \xi_{w_\varphi}$ and

$$\sigma(\mathbf{s}) := \liminf_{k\to+\infty} \frac{1}{[s_{k+1}; s_k, \dots, s_1]}. \tag{2}$$

Note that $\sigma(\mathbf{s}) \leq 1/\gamma$, where $\gamma = [1;1,\cdots] = (1+\sqrt{5})/2$ denotes the golden ratio. Bugeaud and Laurent proved the following result [11, Theorem 3.1].

Theorem 1.1 (Bugeaud-Laurent, 2005)**.** *Let $\varphi = [0; s_1, s_2, \dots] \in [0,1]\setminus\mathbb{Q}$ and let $\sigma = \sigma(\mathbf{s})$. Then*

$$\widehat{\omega}_2(\xi_\varphi) = \widehat{\omega}_2^*(\xi_\varphi) = 2+\sigma, \qquad \widehat{\lambda}_2(\xi_\varphi) = \frac{1+\sigma}{2+\sigma},$$
$$\omega_2(\xi_\varphi) = \omega_2^*(\xi_\varphi) = \frac{2}{\sigma}+1, \qquad \lambda_2(\xi_\varphi) = 1.$$

The set of values taken by $2+1/\sigma(\mathbf{s})$ is called Cassaigne's spectrum (see [14, §4]). It is a compact subset of $[0,+\infty]$ with empty interior. Let us describe shortly the ideas behind the computation of the exponents of ξ_φ. The theory of continued fractions (see for example [35, Chapter I]) ensures that the numerator and denominator of the convergents of ξ_φ are given by the coefficients of the matrices $\Phi(u)$, where u is a prefix of w_φ and $\Phi : \mathcal{A}^* \to \mathcal{M} := \mathrm{Mat}_{2\times 2}(\mathbb{Z}) \cap \mathrm{GL}_2(\mathbb{Q})$ is the morphism of monoids defined by

$$\Phi(a) = \begin{pmatrix} a & 1 \\ 1 & 0 \end{pmatrix} \quad \text{and} \quad \Phi(b) = \begin{pmatrix} b & 1 \\ 1 & 0 \end{pmatrix}. \tag{3}$$

Moreover, when u is a palindrome, the matrix $\Phi(u)$ is symmetric and the mirror formula provides good simultaneous approximations $(p_j/q_j, p_{j-1}/q_j)$ to $(\xi_\varphi, \xi_\varphi^2)$ (see [1] and [2] for other results based on this property). Yet, w_φ has a lot of palindromic prefixes (see [11, Lemma 5.3]). They yield enough explicit simultaneous approximations to ξ_φ, ξ_φ^2 to compute $\lambda_2(\xi_\varphi)$ and $\widehat{\lambda}_2(\xi_\varphi)$. To obtain $\omega_2^*(\xi_\varphi)$ and $\widehat{\omega}_2^*(\xi_\varphi)$, Bugeaud and Laurent consider the quadratic numbers $\alpha_k := [0; w_k w_k \cdots]$ for $k \geq 1$ (see [11, §6]). They are very good approximations to ξ_φ since $w_k^{1+s_{k+1}}$ is a common prefix of w_φ and $w_k w_k \cdots$ (see [11, Lemma 5.2]). Finally, to get the last pair $\omega_2(\xi_\varphi)$, $\widehat{\omega}_2(\xi_\varphi)$, they use the polynomials P_k defined below. They notice that $1/\alpha_k$ is the fixed point of the homography (fractional linear transformation) associated to the matrix $\mathrm{w}_k := \Phi(w_k)$. In particular, setting

$$\mathrm{U}(\mathrm{w}) := -c + (a-d)X + bX^2 \quad \text{for each } \mathrm{w} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{Mat}_{2\times 2}(\mathbb{R}), \tag{4}$$

and defining $P_k := U(\mathrm{w}_k)$, we have $P_k(\alpha_k) = 0$, and $P_k(\xi_\varphi)$ tends to 0 very "quickly".

In a preceding paper [24], we consider instead a general morphism $\Phi : \mathcal{A}^* \to \mathcal{M}$ where $A := \Phi(a)$ and $B := \Phi(b)$ are any matrices of $\mathcal{M}$ such that $\det(AB - BA) \neq 0$ and that the content of $\mathrm{w}_k := \Phi(w_k)$ (defined as the greatest common divisor of the coefficients of w_k) is bounded for $k \geq 0$. Surprisingly, it is still possible to built a sequence of symmetric matrices $(\mathbf{y}_i)_{i\geq 0}$ from $(\mathrm{w}_k)_{k\geq 0}$, which plays the exact same role as the one Laurent and Bugeaud construct from the palindromic prefixes of w_φ (even though A and B are not necessarily symmetric themselves). Under some technical conditions on the growth of $(\mathrm{w}_k)_{k\geq 0}$ and $(\det(\mathrm{w}_k))_{k\geq 0}$, we prove that $(\mathbf{y}_i)_{i\geq 0}$ converges projectively to a symmetric matrix $\begin{pmatrix} 1 & \xi \\ \xi & \xi^2 \end{pmatrix}$. In [24] we further give explicit formulas for the four exponents λ_2, $\widehat{\lambda}_2$, ω_2 and $\widehat{\omega}_2$ associated to ξ when $\mathbf{s}$ is bounded. They generalize the formulas of Theorem 1.1 with the introduction of a second parameter δ associated to the growth of $|\det(\mathrm{w}_k)|$ in addition to σ, see Theorem 1.3. Indeed, compared to the morphism Φ defined by

(3), which yields $\det(\mathrm{w}_k) = \pm 1$ for each k, this new construction provides a larger set of singular points by allowing $|\det(\mathrm{w}_k)|$ to diverge, as Roy did for the Fibonacci word in [30].

In this paper, we complete and extend the results of [24] by considering a general sequence $\mathbf{s}$ (not necessarily bounded) and by relaxing the condition that the content of the matrices $\mathrm{w}_k = \Phi(w_k)$ is bounded, which brings in the delicate question of controlling it. Moreover, we also compute the exponents ω_2^* and $\widehat{\omega}_2^*$ that were left out in our previous study. To do this, we use the following surprising phenomenon. For each $k \geq 0$, write $P_k := \mathrm{U}(\mathrm{w}_k)$ and denote by α_k the root of P_k closest to ξ. Then $P_k(\xi)$ tends to 0 as k tends to infinity, and $(\alpha_k)_{k\geq 0}$ is the sequence of best quadratic approximations to ξ, exactly as in the continued fraction case (although we are working with two general matrices A and B). For each matrix $\mathrm{w} \in \mathrm{Mat}_{2\times 2}(\mathbb{R})$ (resp. polynomial P) we denote by $\|\mathrm{w}\|$ (resp. $H(P)$) the largest absolute value of its coefficients.

Definition 1.2. Let $\mathbf{s} = (s_k)_{k\geq 1}$ be a sequence of positive integers and write $\sigma := \sigma(\mathbf{s})$ as in (2). The set *Sturm*$(\mathbf{s})$ consists of the real numbers ξ which are neither rational nor quadratic, such that there exists a sequence of matrices $(\mathrm{w}_k)_{k\geq 0}$ in $\mathcal{M} := \mathrm{Mat}_{2\times 2}(\mathbb{Z}) \cap \mathrm{GL}_2(\mathbb{Q})$ with the following properties. For each $k \geq 0$, we denote by c_k the content of w_k and we write $\widetilde{\mathrm{w}}_k := c_k^{-1}\mathrm{w}_k$ and $P_k := U(\widetilde{\mathrm{w}}_k)$. Then

(i) $\mathrm{w}_{k+1} = \mathrm{w}_k^{s_{k+1}}\mathrm{w}_{k-1}$ for each $k \geq 1$ and $\det(\mathrm{w}_0\mathrm{w}_1 - \mathrm{w}_1\mathrm{w}_0) \neq 0$.

(ii) There exists $c > 0$ such that $\|\mathrm{w}_k^{\ell+1}\mathrm{w}_{k-1}\| \geq c\|\mathrm{w}_k\|\|\mathrm{w}_k^{\ell}\mathrm{w}_{k-1}\|$ for each k, ℓ with $k \geq 1$ and $0 \leq \ell \leq s_{k+1}$.

(iii) $P_k(\xi)/H(P_k)$ tends to 0 as k tends to infinity.

(iv) $|\det(\widetilde{\mathrm{w}}_k)| \leq \|\widetilde{\mathrm{w}}_k\|^{\sigma/(1+\sigma)+o(1)}$ as k tends to infinity.

We define the set *Sturm* by

$$\mathit{Sturm} := \bigcup_{\mathbf{s}} \mathit{Sturm}(\mathbf{s}).$$

According to our main result below each $\xi \in \mathit{Sturm}(\mathbf{s})$ satisfies $\lambda_2(\xi) \geq 1/(1+\sigma) > 1/2$, and thus is transcendental (see [10, Theorem 2.10]). As we will see *Sturm*$(\mathbf{s})$ is countably infinite (see the remarks after Definition 1.9). It also contains the real numbers ξ_φ associated to $\mathbf{s}$, and Theorem 1.1 follows as a special case of our main result below, taking for granted that $\delta(\xi_\varphi) = 0$.

Theorem 1.3. *Let $\mathbf{s}$ be a sequence of positive integer and set $\sigma := \sigma(\mathbf{s})$. There is a function*

$$\delta : \mathrm{S}\mathit{turm}(\mathbf{s}) \to [0, \sigma/(1+\sigma)],$$

whose image $\Delta(\mathbf{s})$ is a dense subset of $[0, \sigma/(1+\sigma)]$, with the following property. For each $\xi \in$ Sturm$(\mathbf{s})$, writing $\delta := \delta(\xi)$, we have

$$\widehat{\omega}_2(\xi) = \widehat{\omega}_2^*(\xi) = 1 + (1-\delta)(1+\sigma), \qquad \widehat{\lambda}_2(\xi) = \frac{(1-\delta)(1+\sigma)}{1+(1-\delta)(1+\sigma)},$$
$$\omega_2(\xi) = \omega_2^*(\xi) = \frac{2-\delta}{\sigma} + 1 - \delta, \qquad 1-\delta \leq \lambda_2(\xi) \leq \max\Big(1-\delta, \frac{1}{1-\delta+\sigma}\Big).$$

If moreover δ satisfies the stronger condition $\delta < h(\sigma)$, where $h(\sigma) = \sigma/2 + 1 - \sqrt{(\sigma/2)^2+1}$, then

$$\lambda_2(\xi) = 1-\delta \quad \text{and} \quad \widehat{\lambda}_{\min}(\xi) = \frac{(1-\delta)(1+\sigma)}{2+\sigma}.$$

The formula for $\lambda_2(\xi)$ still holds if $\delta = h(\sigma)$.

The exponent $\widehat{\lambda}_{\min}$ above, defined in [25] and related to work of Fischler [20], satisfies $\widehat{\lambda}_{\min} \leq \widehat{\lambda}$. We recall its definition in the next section.

If $\mathbf{s}$ is unbounded, then $\sigma(\mathbf{s})$ is equal to 0 by definition. Theorem 1.3 implies that for each $\xi \in \mathit{Sturm}(\mathbf{s})$, we have $\delta(\xi) = 0 = h(0)$ and

$$\widehat{\omega}_2(\xi) = \widehat{\omega}_2^*(\xi) = 2, \quad \omega_2(\xi) = \omega_2^*(\xi) = \infty, \quad \widehat{\lambda}_2(\xi) = \frac{1}{2} \quad \text{and} \quad \lambda_2(\xi) = 1.$$

Recall that the spectrum of an exponent ν is the set of its values $\mathrm{spec}(\nu) := \nu(\mathbb{R} \setminus \overline{\mathbb{Q}})$. The spectrum of ω_2, resp. ω_2^*, is equal to $[2, +\infty]$ by a result of Bernik [7], resp. Baker and Schmidt [5]. We also have $\mathrm{spec}(\lambda_2) = [1/2, +\infty]$ by [6] and [38]. However, the spectrum of the uniform exponents is more mysterious and complicated. Let $\mathbf{1} = (s_k)_{k\geq 1}$ denote the constant sequence $s_k = 1$ for each $k \geq 1$. Its associated Sturmian characteristic word is the Fibonacci word. The set $\mathit{Sturm}(\mathbf{1})$ is of particular interest, since it contains Roy's extremal numbers [28] and Fibonacci type numbers [30]. We have $\sigma(\mathbf{1}) = 1/\gamma$, where $\gamma = (1 + \sqrt{5})/2$ is the golden ratio, and the set $\Delta(\mathbf{1})$ is dense in $[0, 1/\gamma^2]$. From this we recover the result of Roy according to which the spectrum of $\widehat{\lambda}_2$ and that of $\widehat{\omega}_2$ are dense in $[1/2, 1/\gamma]$ and in $[2, \gamma^2]$ respectively [30]. Our first corollary follows by applying Theorem 1.3 to the sequence $\mathbf{1}$.

Corollary 1.4. *The spectrum of $\widehat{\omega}_2^*$ contains a dense subset of the interval $[2, \gamma^2]$.*

Theorem 1.1 gives $2 + \sigma(\mathbf{s}) \in \mathrm{spec}(\widehat{\omega}_2^*)$ for each $\mathbf{s}$. Nonetheless, the set of values taken by $2+\sigma(\mathbf{s})$ is a compact subset of $[2, \gamma^2]$ with empty interior, and thus far from being dense (see the paper of Cassaigne [14]). Also note that a theorem of Bugeaud (see [8] and [10, Theorem 5.6]) shows that the full interval $[1, 3/2]$ is contained in the spectrum of $\widehat{\omega}_2^*$.

Although it is possible to have $\omega_2^*(\xi) < \omega_2(\xi)$ (see [10, Theorem 5.7] and [9] for explicit examples), our next corollary, proven in Section 5.4, shows that it does not happen if $\widehat{\omega}_2(\xi)$ is sufficiently close to its maximal value γ^2. Note that as far as the author know, we do not know if there exists a real number ξ with $\widehat{\omega}_2^*(\xi) < \widehat{\omega}_2(\xi)$.

Corollary 1.5. *There exists $\varepsilon > 0$ with the following property. Let ξ be a real number which is neither rational nor quadratic. If $\widehat{\omega}_2(\xi) > \gamma^2 - \varepsilon$, then $\xi \in \mathit{Sturm}(\mathbf{1})$. In particular*

$$\widehat{\omega}_2^*(\xi) = \widehat{\omega}_2(\xi) \quad \text{and} \quad \omega_2^*(\xi) = \omega_2(\xi),$$

and the set $\mathrm{spec}(\widehat{\omega}_2^*) \cap [\gamma^2 - \varepsilon, \gamma^2]$ is countably infinite.

Applying Theorem 1.3 to the sequence $\mathbf{1}$ we also deduce new information on $\mathrm{spec}(\widehat{\lambda}_{\min})$.

Corollary 1.6. *The spectrum of $\widehat{\lambda}_{\min}$ contains a dense subset of the interval $[\kappa, 1/\gamma]$, where*

$$\kappa := \frac{1 - h(1/\gamma)}{\gamma} = 0.4558\cdots$$

In this paper, we will use an equivalent definition of the set $\mathit{Sturm}(\mathbf{s})$ (see Section 5.4), which connects to a class of numbers considered by Fischler in [20]. Before stating it, let us go back to the combinatorial properties of the Sturmian characteristic word w_φ. If $u, v, w \in \mathcal{A}^*$ satisfy $u = vw$, we write $v^{-1}u = w$. If u has length at least 2, then we denote by u' the word u deprived of its two last letters. By [11, Lemma 5.3], the sequence $(\pi_i)_{i\geq 0}$ of palindromic prefixes of m_φ (ordered by increasing length) consists of $b, \dots, b^{s_1-1}$ and the words

$$(w_k^{\ell+1} w_{k-1})' \quad \text{with } k \geq 1 \text{ and } 0 \leq \ell < s_{k+1}.$$

Moreover, there exists a function $\psi := \psi_{\mathbf{s}}$ defined over $\mathbb{N}$ (with $\psi(i) < i$ for each $i \in \mathbb{N}$) such that

$$\pi_{i+1} = \pi_i(\pi_{\psi(i)}^{-1}\pi_i) \tag{5}$$

for each large enough i (see [19, §3]). The precise definition of $\psi_{\mathbf{s}}$ is given in the next section (see Definition 2.1). In the Fibonacci case where $\mathbf{s} = \mathbf{1}$, the associated function ψ satisfies $\psi(i) = i - 2$ for each i. In [20], Fischler studied real numbers ξ_w associated to words w with a large density of palindromic prefixes (also see [19]). He introduced a new Diophantine exponent β_0 (whose definition is recalled in the next section) which is closely related to $\widehat{\lambda}_2$, and was able to compute $\beta_0(\xi_w)$ and to give a complete description of the set $\beta_0(\mathbb{R}\setminus\overline{\mathbb{Q}}) \cap (1, 2)$. Our primary motivation to introduce the new class of numbers *Sturm*($\mathbf{s}$) comes from the following result (a combination of [20, Theorem 4.1] with [19, Lemma 7.1]), where we identify $\mathbb{R}^3$ with the space of matrices $\mathrm{Mat}_{2\times 2}(\mathbb{R})$ under the map

$$(x_0, x_1, x_2) \longmapsto \begin{pmatrix} x_0 & x_1 \\ x_1 & x_2 \end{pmatrix}, \tag{6}$$

and we denote by $\mathrm{Adj}(\mathrm{w})$ the adjoint of a matrix $\mathrm{w} \in \mathrm{Mat}_{2\times 2}(\mathbb{R})$.

Theorem 1.7 (Fischler, 2007)**.** *Let ξ be a real number with $\beta_0(\xi) < 2$, which is neither rational nor quadratic. Then, there exists a sequence $(\mathbf{v}_i)_{i\geq 0}$ of non-zero primitive points in $\mathbb{Z}^3$ (identified with the corresponding symmetric matrices) with the following properties. The sequence $(\|\mathbf{v}_i\|)_{i\geq 0}$ tends to infinity,*

$$\|\mathbf{v}_i \wedge \Xi\| = \|\mathbf{v}_i\|^{-1+o(1)}, \tag{7}$$

where $\Xi := (1, \xi, \xi^2)$, and there exists a function $\psi : \mathbb{N} \to \mathbb{N}$ such that $\mathbf{v}_{i+1}$ is collinear to $\mathbf{v}_i\mathrm{Adj}(\mathbf{v}_{\psi(i)})\mathbf{v}_i$ for each large enough i. If moreover $\beta_0(\xi) < \sqrt{3}$, then we may choose $\psi = \psi_{\mathbf{s}}$ for a bounded sequence $\mathbf{s}$ of positive integers.

The result of Fischler is more precise. It shows that ψ belongs to a narrow class of functions called *asymptotically reduced* (see [20, Definition 2.1]), which includes all functions $\psi_{\mathbf{s}}$ with $\mathbf{s}$ bounded. In general each asymptotically reduced function comes from an infinite word with a large density of palindromic prefixes π_i in such a way that the recurrence (5) holds [19, Section 3.1]. In [20], Fischler motivates and asks the following question.

Problem. Let $\xi \in \mathbb{R}$ which is neither rational nor quadratic. Does the condition $\beta_0(\xi) < 2$ imply $\widehat{\lambda}_2(\xi) = 1/\beta_0(\xi)$?

The following partial answer proves a claim made by Fischler in [18].

Theorem 1.8. *Let ξ be a real number which is neither rational nor quadratic. If $\beta_0(\xi) < \sqrt{3}$, then $\widehat{\lambda}_2(\xi) = \widehat{\lambda}_{\min}(\xi) = 1/\beta_0(\xi)$.*

The idea is to prove that the condition $\beta(\xi) < \sqrt{3}$ implies that $\xi \in$ *Sturm*($\mathbf{s}$) and $\delta(\xi) = 0$, where $\mathbf{s}$ is the sequence given by Theorem 1.7, and $\delta(\xi)$ is the quantity appearing in Theorem 1.3 (see Section 5.4). The first observation follows relatively easily from the following alternative definition of the set *Sturm*($\mathbf{s}$) (see Section 5.4).

Definition 1.9. Let $\mathbf{s}$ be a sequence of positive integers, write $\sigma := \sigma(\mathbf{s})$ and $\psi = \psi_{\mathbf{s}}$. The set *Sturm*($\mathbf{s}$) is the set of real numbers ξ which are neither rational nor quadratic, and for which there exists a sequence $(\mathbf{y}_i)_{i\geq 0}$ of non-zero primitive points in $\mathbb{Z}^3$ (identified with their symmetric matrices) with the following properties.

(i) The sequence $(\mathbf{y}_i)_{i\geq 0}$ converges projectively to $\Xi := (1, \xi, \xi^2)$.
(ii) The matrix $\mathbf{y}_{i+1}$ is proportional to $\mathbf{y}_i\mathrm{Adj}(\mathbf{y}_{\psi(i)})\mathbf{y}_i$ for each large enough i.
(iii) We have $|\det(\mathbf{y}_i)| \leq \|\mathbf{y}_i\|^{\sigma/(1+\sigma)+o(1)}$ as i tends to infinity with $\psi(i+1) < i$.

We end this introduction with a few remarks, see Section 4.3 for more details.

Remarks. Since $\xi \notin \mathbb{Q}$, the sequence $(\|\mathbf{y}_i\|)_{i\geq 0}$ tends to infinity and $\det(\mathbf{y}_i) \neq 0$ for i large enough (since for large i, by (ii) and (i), the kernel of the matrix $\mathbf{y}_i$ is included in the kernel of Ξ)

The parameter $\delta(\xi)$ in Theorem 1.3 can be defined as the supremum limit of $\log|\det(\mathbf{y}_i)|/\log\|\mathbf{y}_i\|$ as i tends to infinity with $\psi(i+1) < i$.

In view of (5), if $\xi = \xi_\varphi$, then we can take $\mathbf{y}_i = \Phi(\pi_i)$ for i large, where Φ is as in (3). Since in that case $\det(\mathbf{y}_i) = \pm 1$, we have $\delta(\xi_\varphi) = 0$.

By (ii), the sequence $(\mathbf{y}_i)_{i\geq i_0}$ (with i_0 large enough) is entirely determined by three points in $\mathbb{Z}^3$. We thus have a surjection $(\mathbb{Z}^3)^3 \to$ *Sturm*($\mathbf{s}$), and *Sturm*($\mathbf{s}$) is therefore at most countable.

It follows easily from [24, Definition 6.2] that the set of Sturmian type numbers constructed in [24] is included in *Sturm*. More precisely, if $\mathbf{s}$ is bounded, then the (infinite) set of proper $\psi_{\mathbf{s}}$-Sturmian numbers (see Definition 4.12 and [24, Proposition 6.1]) is included in *Sturm*$(\mathbf{s})$. This yields the density of $\Delta(\mathbf{s})$ in $[0, \sigma/(1+\sigma)]$ when $\sigma = \sigma(\mathbf{s}) > 0$.

Our paper is organized as follows. In the next section, we define the Diophantine exponents involved in Theorem 1.3 and introduce some notation. Sections 3 and 4 are devoted to the theory of Sturmian sequences of matrices; in the former we focus on the combinatorial aspects and establish new key-identities, while in the latter we study the asymptotic behavior of these sequences. Combining those results, we obtain a new characterization of the set *Sturm* in terms of Sturmian sequences of matrices (see Section 4.3). This allows us, using parametric geometry of numbers, to prove our main theorem in the last section.

2. Notation

Given a positive integer n and $\mathbf{x} \in \mathbb{R}^n$, we define its norm $\|\mathbf{x}\|$ as the largest absolute value of its coordinates. Let ξ be a real number which is neither rational nor quadratic. We associate to ξ several classical Diophantine exponents as follows. The ordinary (resp. uniform) exponent of simultaneous approximation $\lambda_2(\xi)$, (resp. $\widehat{\lambda}_2(\xi)$) is the supremum of real numbers λ such that, for arbitrarily large values of X (resp. for each X large enough), there exists $\mathbf{x} \in \mathbb{Z}^3 \setminus \{0\}$ satisfying

$$\|\mathbf{x} \wedge \Xi\| \leq X^{-\lambda} \quad \text{and} \quad \|\mathbf{x}\| \leq X,$$

where $\mathbf{x} \wedge \mathbf{y} \in \mathbb{R}^3$ denotes the cross product of $\mathbf{x}$ and $\mathbf{y}$ in $\mathbb{R}^3$. Similarly, the ordinary (resp. uniform) exponent $\omega_2(\xi)$, (resp. $\widehat{\omega}_2(\xi)$) is the supremum of real numbers ω such that, for arbitrarily large values of X (resp. for each X large enough), there exists $\mathbf{x} \in \mathbb{Z}^3 \setminus \{0\}$ satisfying

$$|\mathbf{x} \cdot \Xi| \leq X^{-\omega} \quad \text{and} \quad \|\mathbf{x}\| \leq X,$$

where $\mathbf{x} \cdot \mathbf{y}$ denotes the standard scalar product of $\mathbf{x}$ and $\mathbf{y}$ in $\mathbb{R}^3$. The exponent $\omega_2^*(\xi)$ (resp. $\widehat{\omega}_2^*(\xi)$) is the supremum of real numbers ω such that, for arbitrarily large values of X (resp. for each X large enough), there is an algebraic number α of degree at most 2 satisfying

$$0 < |\xi - \alpha| \leq H(\alpha)^{-1} X^{-\omega} \quad \text{and} \quad H(\alpha) \leq X,$$

where $H(\alpha)$ is the *height* of α, defined as the largest absolute value of the coefficients of its irreducible minimal polynomial over $\mathbb{Z}$. See [11] for the motivation of the division by $H(\alpha)$ in the left-hand side. We now recall the definitions of the last two exponents $\beta_0(\xi)$ and $\widehat{\lambda}_{\min}(\xi)$, introduced respectively by Fischler in [20] and by the author in [25] on the basis of [20]. Set $\Xi = (1, \xi, \xi^2)$ and for each $0 \leq \mu < \lambda(\xi)$, denote by $\widehat{\lambda}_\mu(\Xi)$ the supremum of the real numbers λ for which

$$\|\mathbf{x} \wedge \Xi\| \leq \min(X^{-\lambda}, \|\mathbf{x}\|^{-\mu}) \quad \text{and} \quad \|\mathbf{x}\| \leq X$$

admits a non-zero integer solution for each sufficiently large value of X. The map $\mu \mapsto \widehat{\lambda}_\mu(\Xi)$ is non-increasing, and for $\mu = 0$ we simply have $\widehat{\lambda}_0(\Xi) = \widehat{\lambda}_2(\xi)$. We set

$$\beta_0(\xi) = \begin{cases} \lim_{\mu \to 1^-} 1/\widehat{\lambda}_\mu(\Xi) & \text{if } \lambda_2(\xi) \geq 1, \\ \infty & \text{else} \end{cases}, \quad \text{and} \quad \widehat{\lambda}_{\min}(\xi) = \lim_{\mu \to \lambda_2(\xi)^-} \widehat{\lambda}_\mu(\Xi).$$

In particular $\widehat{\lambda}_{\min}(\xi) \leq \widehat{\lambda}_2(\xi)$. Note that the definition of $\widehat{\lambda}_{\min}$ in [25] applies to general points $\Xi = (1, \xi, \eta) \in \mathbb{R}^3$. In the current situation, the two above exponents are connected in the following way: if $\beta_0(\xi) < 2$, then $\lambda_2(\xi) = 1$ and $\widehat{\lambda}_{\min}(\xi) = 1/\beta_0(\xi)$ (see [25, Lemma 1.3]). The classical general estimates below are valid for each $\xi \in \mathbb{R}$ which is neither rational nor quadratic. Recall that $\gamma = (1+\sqrt{5})/2$ denotes the golden ratio. First

$$\frac{1}{2} \leq \widehat{\lambda}_2(\xi) \leq 1/\gamma \quad \text{and} \quad 2 \leq \widehat{\omega}_2(\xi) \leq \gamma^2.$$

The lower bounds are obtained by the Dirichlet box principle, the upper bounds follow respectively from [15, Theorem 1*a*] and from [4]. Jarník's identity [21, Theorem 1] links $\widehat{\lambda}_2(\xi)$ and $\widehat{\omega}_2(\xi)$ as follows

$$\widehat{\lambda}_2(\xi) = 1 - \frac{1}{\widehat{\omega}_2(\xi)}. \tag{8}$$

We also have (see [10, Theorem 2.5])

$$1 \leq \widehat{\omega}_2^*(\xi) \leq \min\{\omega_2^*(\xi), \widehat{\omega}_2(\xi)\} \leq \max\{\omega_2^*(\xi), \widehat{\omega}_2(\xi)\} \leq \omega_2(\xi). \tag{9}$$

We now recall the notion of *Sturmian functions* $\psi_{\mathbf{s}}$, which intervene in the recurrence relation (5) of the palindromic prefixes of a Sturmian characteristic word. They play a central role in [24] (see also [20] and [19]).

Definition 2.1. Let $\mathbf{s} = (s_k)_{k\geq 1}$ be a sequence of positive integers and for each $k \geq 0$ set $t_k = s_0 + s_1 + \cdots + s_k$ (where $s_0 = -1$). We associate to $\mathbf{s}$ a function $\psi = \psi_{\mathbf{s}}$ defined on $\mathbb{N}$ as follows.

$$\psi(i) := \begin{cases} t_{k-1} - 1 & \text{if } i = t_k \text{ with } k \geq 1, \\ i - 1 & \text{else.} \end{cases}$$

Note that $\mathbf{s}$ is entirely characterised by ψ, the sequence $(t_k)_{k\geq 1}$ consisting of the integers n such that $\psi(n) \leq n - 2$.

We denote by $\|\mathrm{w}\|$ the norm of a matrix $\mathrm{w} \in \mathrm{Mat}_{2\times 2}(\mathbb{R})$ defined as the largest absolute value of its coefficients. Recall that $\mathbb{R}^3$ is identified with $\mathrm{Mat}_{2\times 2}(\mathbb{R})$ under the map (6). Accordingly, we define the determinant $\det(\mathbf{x}) = x_0 x_2 - x_1^2$ of a point $\mathbf{x} = (x_0, x_1, x_2) \in \mathbb{R}^3$. Similarly, given symmetric matrices $\mathbf{x}, \mathbf{y}$, we write $\mathbf{x} \wedge \mathbf{y}$ to denote the cross product of the corresponding points in $\mathbb{R}^3$. We also identify $\mathbb{R}^3$ (and thus $\mathrm{Mat}_{2\times 2}(\mathbb{R})$) to $\mathbb{R}[X]_{\leq 2}$, the space of polynomial of degree at most 2, via the map $(x_0, x_1, x_2) \longmapsto x_0 + x_1 X + x_2 X^2$.
For any $\mathrm{w} \in \mathrm{Mat}_{2\times 2}(\mathbb{R})$, we denote by ${}^t\mathrm{w}$ its transpose, and by $\mathrm{Adj}(\mathrm{w})$ its adjoint. The *content* of a non-zero matrix $\mathrm{w} \in \mathrm{Mat}_{2\times 2}(\mathbb{Z})$ or of a non-zero point $\mathbf{y} \in \mathbb{Z}^3$ is the greatest common divisor of its coefficients. We say that such a matrix or point is *primitive* if its content is 1. More generally, if $\mathrm{w} \in \mathrm{Mat}_{2\times 2}(\mathbb{R}) \setminus \{0\}$ is proportional to a matrix of $\mathrm{Mat}_{2\times 2}(\mathbb{Z})$, we say that w is *defined over* $\mathbb{Q}$ and we denote by $\mathrm{cont}(\mathrm{w})$ the positive real number α such that $\alpha^{-1}\mathrm{w}$ is a primitive matrix of $\mathrm{Mat}_{2\times 2}(\mathbb{Z})$. We set

$$J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \quad \text{and} \quad \mathrm{Id} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}. \tag{10}$$

Given a non-empty interval $A \subseteq \mathbb{R}$, a positive integer n and $F : A \to \mathbb{R}^n$, we denote by $\|F\|_\infty := \sup_{q\in A} \|F(q)\|$. Finally, let I be a set (typically of the form $\mathbb{N}^r$), $(a_{\underline{i}})_{\underline{i}\in I}$ and let $(b_{\underline{i}})_{\underline{i}\in I}$ be two sequences of non-negative real numbers indexed by I. For any non-empty subset $J \subseteq I$, we write '$a_{\underline{i}} \ll b_{\underline{i}}$ for $\underline{i} \in J$' or '$b_{\underline{i}} \gg a_{\underline{i}}$ for $\underline{i} \in J$' if there is a constant $c > 0$ such that for each $\underline{i} \in J$ we have $a_{\underline{i}} \leq c b_{\underline{i}}$. We write '$a_{\underline{i}} \asymp b_{\underline{i}}$ for $\underline{i} \in J$' if both $a_{\underline{i}} \ll b_{\underline{i}}$ and $b_{\underline{i}} \ll a_{\underline{i}}$ for $\underline{i} \in J$ hold. In the special case where $I = \mathbb{N}$, unless otherwise stated, we will always implicitly take J of the form $[j_0, +\infty) \cap \mathbb{N}$ for j_0 large enough, and we will simply write $a_{\underline{i}} \ll b_{\underline{i}}$, $b_{\underline{i}} \gg a_{\underline{i}}$ and $a_{\underline{i}} \asymp b_{\underline{i}}$.

3. Combinatorics of Sturmian sequences of matrices

Let $\mathbf{s} = (s_k)_{k\geq 1}$ be a sequence of positive integers (not necessarily bounded) and set $\psi = \psi_{\mathbf{s}}$ (see Definition 2.1). We define below the notions of ψ-Sturmian sequences and admissible ψ-Sturmian sequences of matrices. We develop the latter notion in §3.1. To an admissible ψ-Sturmian sequence correspond two sequences of symmetric matrices $(\mathbf{y}_i)_i$ and $(\mathbf{z}_i)_i$, which we also view as sequences in

$\mathbb{R}^3$ (see §3.2). In our applications, $(\mathbf{y}_i)_i$ will provide "good" solutions to the problem of simultaneous approximation, whereas $(\mathbf{z}_i)_i$ will be related to the problem with polynomials. In §3.3, we establish a new and surprising formula for $(\mathbf{z}_i)_i$. This is one of the key-properties for studying the exponents ω_2^* and $\widehat{\omega}_2^*$.

Definition 3.1. A *ψ-Sturmian* sequence in $\mathrm{GL}_2(\mathbb{R})$ is a sequence $(\mathrm{w}_k)_{k\geq 0}$ such that $\mathrm{w}_0, \mathrm{w}_1 \in \mathrm{GL}_2(\mathbb{R})$, and for each $k \geq 1$, we have the recurrence relation $\mathrm{w}_{k+1} = \mathrm{w}_k^{s_{k+1}}\mathrm{w}_{k-1}$.

Clearly, such a sequence is entirely determined by its first two elements w_0 and w_1.

Definition 3.2. Let $(\mathrm{w}_k)_{k\geq 0}$ be a ψ-Sturmian sequence in $\mathrm{GL}_2(\mathbb{R})$.

- We say that $(\mathrm{w}_k)_{k\geq 0}$ is *admissible* if the matrix $\mathrm{w}_0\mathrm{w}_1 - \mathrm{w}_1\mathrm{w}_0$ is invertible. Given an integer $k \geq 1$, the identity $\mathrm{w}_k\mathrm{w}_{k+1} - \mathrm{w}_{k+1}\mathrm{w}_k = -\mathrm{w}_k^{s_{k+1}-1}(\mathrm{w}_{k-1}\mathrm{w}_k - \mathrm{w}_k\mathrm{w}_{k-1})$ implies that $(\mathrm{w}_k)_{k\geq 0}$ is admissible if and only if $\mathrm{w}_{k-1}\mathrm{w}_k - \mathrm{w}_k\mathrm{w}_{k-1}$ is invertible.
- The sequence $(\mathrm{w}_k)_{k\geq 0}$ has a *multiplicative growth* if $\|\mathrm{w}_k^{\ell+1}\mathrm{w}_{k-1}\| \asymp \|\mathrm{w}_k\|\|\mathrm{w}_k^{\ell}\mathrm{w}_{k-1}\|$ for $k \geq 1$ and $0 \leq \ell \leq s_{k+1}$
- Finally, $(\mathrm{w}_k)_{k\geq 0}$ is *defined over* $\mathbb{Q}$ if, for each $k \geq 0$, the matrix w_k is proportional to a matrix of $\mathrm{Mat}_{2\times 2}(\mathbb{Q})$.

3.1. **Admissible sequences.**

In [30] and [24] a ψ-Sturmian sequence $(\mathrm{w}_i)_{i\geq 0}$ is said to be admissible if there exists a matrix $N \in \mathrm{GL}_2(\mathbb{R})$ satisfying

$$\mathrm{w}_0{}^tN, \quad \mathrm{w}_1 N, \quad \text{and} \quad \mathrm{w}_1\mathrm{w}_0{}^tN \quad \text{are symmetric,} \tag{11}$$

which gives a slightly different notion than ours. Note that by taking the transpose of $\mathrm{w}_1\mathrm{w}_0{}^tN$ and using successively the fact that $\mathrm{w}_0{}^tN$ and w_1N are symmetric, (11) implies that

$$\mathrm{w}_1\mathrm{w}_0{}^tN = \mathrm{w}_0\mathrm{w}_1N. \tag{12}$$

In that case and if N is symmetric, then w_0 and w_1 commute; this is a degenerate situation that we want to avoid. According to the next result, if w_0 and w_1 do not commute, then our definition of admissibility is equivalent to the existence of $N \in \mathrm{GL}_2(\mathbb{R})$ satisfying (11). In addition, up to a multiplicative constant, we provide a simple expression for N.

Proposition 3.3. *Let* $\mathrm{w}_0, \mathrm{w}_1 \in \mathrm{GL}_2(\mathbb{R})$*. Then the following conditions are equivalent:*

(i) $\det(\mathrm{w}_0\mathrm{w}_1 - \mathrm{w}_1\mathrm{w}_0) \neq 0$

(ii) $\mathrm{w}_1\mathrm{w}_0 \neq \mathrm{w}_0\mathrm{w}_1$ *and there exists a matrix* $N \in \mathrm{GL}_2(\mathbb{R})$ *satisfying* (11)*.*

If these conditions are satisfied, then any $N \in \mathrm{Mat}_{2\times 2}(\mathbb{R})$ *satisfying* (11) *is proportional to*

$$M = (\mathrm{Id} - \mathrm{w}_1^{-1}\mathrm{w}_0^{-1}\mathrm{w}_1\mathrm{w}_0)J. \tag{13}$$

Proof. Note that the matrix M defined in (13) is invertible if and only if (i) holds. Suppose (i). Since $J\mathbf{x}J = -\det(\mathbf{x})^t\mathbf{x}^{-1}$ for each $\mathbf{x} \in \mathrm{GL}_2(\mathbb{R})$, we have

$$^t\big((\mathrm{Id} - \mathbf{x}^{-1}\mathbf{y}^{-1}\mathbf{x}\mathbf{y})J\big) = -(\mathrm{Id} - \mathbf{y}^{-1}\mathbf{x}^{-1}\mathbf{y}\mathbf{x})J \tag{14}$$

for any $\mathbf{x}, \mathbf{y} \in \mathrm{GL}_2(\mathbb{R})$. In particular ${}^tM = (\mathrm{Id} - \mathrm{w}_0^{-1}\mathrm{w}_1^{-1}\mathrm{w}_0\mathrm{w}_1)J$. Moreover, since $A \in \mathrm{Mat}_{2\times 2}(\mathbb{R})$ is symmetric if and only if $\mathrm{Tr}(AJ) = 0$, using (13) and the above expression of ${}^tM$, it is easily seen that (11) is satisfied with $N = M$, hence (ii).

Now we prove (ii) $\Rightarrow$ (i). First, note that if $N \in \mathrm{GL}_2(\mathbb{R})$ satisfies (11), then we have (12). Since by hypothesis $\mathrm{w}_1\mathrm{w}_0 \neq \mathrm{w}_0\mathrm{w}_1$, it implies that N is not symmetric. Thus $\det({}^tN - N) \neq 0$ and the matrix $(\mathrm{w}_0\mathrm{w}_1 - \mathrm{w}_1\mathrm{w}_0){}^tN = \mathrm{w}_0\mathrm{w}_1({}^tN - N)$ is invertible, hence (i).

Suppose now that (i) and (ii) are satisfied and let us prove the last part of the proposition. In general, the conditions (11) represent a system of three linear equations in the four unknown coefficients of N. By (12), the condition (i) implies that there is no non-zero symmetric matrix $N \in \mathrm{Mat}_{2\times 2}(\mathbb{R})$ solution of this system. Its rank is thus equal to 3 and the space of solution has dimension 1. ∎

3.2. Symmetric matrices associated to Sturmian sequences.

We associate to any admissible ψ-Sturmian sequence two sequences of symmetric matrices $(\mathbf{y}_i)_{i\geq -2}$ and $(\mathbf{z}_i)_{i\geq -1}$ as in [24, Definitions 3.5 and 4.2]. They play a major role in our study.

Definition 3.4. Let $(\mathrm{w}_k)_{k\geq 0}$ be a ψ-Sturmian sequence in $\mathrm{GL}_2(\mathbb{R})$, and let $N \in \mathrm{GL}_2(\mathbb{R})$ be such that $(\mathbf{y}_{-2}, \mathbf{y}_{-1}, \mathbf{y}_0) := (\mathrm{w}_0{}^tN, \mathrm{w}_1 N, \mathrm{w}_1\mathrm{w}_0{}^tN)$ is a triple of symmetric matrices. We define $\mathrm{w}_{-1} = \mathrm{w}_0^{-1}\mathrm{w}_1$, and for each integers $k, \ell \geq 0$ with $0 \leq \ell < s_{k+1}$, we set

$$\mathbf{y}_{t_k+\ell} = \mathrm{w}_k^{\ell+1}\mathrm{w}_{k-1}N_k \quad \text{and} \quad \mathbf{z}_{t_k+\ell} = \frac{1}{\det(\mathrm{w}_k)}\mathbf{y}_{\psi(t_{k+1})} \wedge \mathbf{y}_{t_k+\ell}, \tag{15}$$

where $N_k = N$ if k is even, $N_k = {}^tN$ if k is odd. By [24, Proposition 3.6] the matrix $\mathbf{y}_i$ is symmetric for each $i \geq -2$, so that the wedge product defining $\mathbf{z}_{t_k+\ell}$ makes sense. Note that the left-hand side of (15) remains valid for $\ell = s_{k+1}$. In particular, for each $k \geq 1$, we have

$$\mathbf{y}_{\psi(t_k)} = \mathrm{w}_{k-1}N_k. \tag{16}$$

Remark. According to Proposition 3.3, if $(\mathrm{w}_k)_{k\geq 0}$ is admissible, then the matrix N is proportional to $(\mathrm{Id} - \mathrm{w}_1^{-1}\mathrm{w}_0^{-1}\mathrm{w}_1\mathrm{w}_0)J$. In the following, if we refer to the sequences $(\mathbf{y}_i)_{i\geq -2}$ and $(\mathbf{z}_i)_{i\geq -1}$ associated to an admissible ψ-Sturmian sequence without further precision on N, we will always implicitly take $N = (\mathrm{Id} - \mathrm{w}_1^{-1}\mathrm{w}_0^{-1}\mathrm{w}_1\mathrm{w}_0)J$ in (15).

Those two sequences satisfy a lot of combinatorial properties, for example [24, Eq. (3.4)] yields:

$$\mathbf{y}_{i+1} = \mathbf{y}_i\mathbf{y}_{\psi(i)}^{-1}\mathbf{y}_i \quad (i \geq 0). \tag{17}$$

In the next lemma, we study the degenerate situation where N is symmetric (this is one of the reason why we want to avoid this situation).

Lemma 3.5. *Let $(\mathrm{w}_k)_{k\geq 0}$, $N \in \mathrm{GL}_2(\mathbb{R})$ and $(\mathbf{y}_i)_{i\geq -2}$ be as in Definition 3.4. Then, for each $i \geq -1$ which is not among the t_k ($k \geq 0$), the poins $\mathbf{y}_{i-1}, \mathbf{y}_i, \mathbf{y}_{i+1}$ are linearly dependent. Moreover, the following assertions are equivalent:*

(i) *$(\mathrm{w}_k)_{k\geq 0}$ is not admissible;*

(ii) *N is symmetric;*

(iii) *There is $k \geq 0$ such that $\mathbf{y}_{t_k-1}, \mathbf{y}_{t_k}, \mathbf{y}_{t_k+1}$ are linearly dependent;*

(iv) *The space generated by $(\mathbf{y}_i)_{i\geq -2}$ has dimension at most 2;*

Proof. Eq. (2.1) of [28] combine with $J\mathbf{y}J = -\det(\mathbf{y})\mathbf{y}^{-1}$ (valid for all symmetric matrix $\mathbf{y} \in \mathrm{GL}_2(\mathbb{R})$) gives the identity $\det(\mathbf{x}, \mathbf{y}, \mathbf{z}) = -\det(\mathbf{y})\mathrm{Tr}(J\mathbf{x}\mathbf{y}^{-1}\mathbf{z})$ for all symmetric matrices $\mathbf{x}, \mathbf{y}, \mathbf{z} \in \mathrm{GL}_2(\mathbb{R})$ (also viewed as points in $\mathbb{R}^3$). Since $\mathrm{Tr}(JA) = 0$ if and only if A is symmetric, we obtain the following useful criterion, valid for each symmetric matrices $\mathbf{x}, \mathbf{y}, \mathbf{z} \in \mathrm{GL}_2(\mathbb{R})$:

$$\det(\mathbf{x}, \mathbf{y}, \mathbf{z}) = 0 \Leftrightarrow \mathbf{x}\mathbf{y}^{-1}\mathbf{z} \text{ is symmetric.} \tag{18}$$

Now, let $i \geq -1$ be an index not among the t_k. Then $\psi(i) = i - 1$, and by (17) the matrix $\mathbf{y}_i\mathbf{y}_{i+1}^{-1}\mathbf{y}_{i-1} = \mathbf{y}_{i-1}\mathbf{y}_i^{-1}\mathbf{y}_{i-1}$ is symmetric. We deduce from (18) that $\det(\mathbf{y}_i, \mathbf{y}_{i+1}, \mathbf{y}_{i-1}) = 0$. This proves the first part of our lemma.

(i) $\Leftrightarrow$ (ii) by (12) and Proposition 3.3. We obtain (ii) $\Leftrightarrow$ (iii) by noticing that if $i = t_k$ with $k \geq 0$, then $\mathbf{y}_{i+1} = \mathrm{w}_k\mathbf{y}_i$, $\mathbf{y}_{i-1} = \mathbf{y}_{\psi(t_{k+1})} = \mathrm{w}_kN_{k+1}$, so that $\mathbf{y}_i\mathbf{y}_{i+1}^{-1}\mathbf{y}_{i-1} = N_{k+1}$, and $\det(\mathbf{y}_{t_k}, \mathbf{y}_{t_k+1}, \mathbf{y}_{t_k-1}) = 0$ if and only if N is symmetric. Lastly, we get (iii) $\Leftrightarrow$ (iv) by combining the above combined with the first part of the lemma. $\square$

We now prove that any sequence satisfying (17) comes from a ψ-Sturmian sequence. This will play a crucial role in establishing the new characterization of the set *Sturm* in Section 4.3.

Proposition 3.6. *Let $i_0 \geq -2$ be an integer and let $(\mathbf{v}_i)_{i\geq -2}$ be a sequence of symmetric matrices such that $\det(\mathbf{v}_i) \neq 0$ for each $i \geq -2$, and*

$$\mathbf{v}_{i+1} = \mathbf{v}_i \mathbf{v}_{\psi(i)}^{-1} \mathbf{v}_i \tag{19}$$

for each $i \geq 0$. Then, there are $N \in \mathrm{GL}_2(\mathbb{R})$ and a ψ-Sturmian sequence $(\mathrm{w}_k)_{k\geq 0}$ with the following property. We have $(\mathbf{v}_{-2}, \mathbf{v}_{-1}, \mathbf{v}_0) := (\mathrm{w}_0{}^tN, \mathrm{w}_1 N, \mathrm{w}_1\mathrm{w}_0{}^tN)$, and $(\mathbf{v}_i)_{i\geq -2}$ is precisely the sequence $(\mathbf{y}_i)_{i\geq -2}$ associated to $(\mathrm{w}_k)_{k\geq 0}$ and N by Definition 3.4. If moreover the space generated by $(\mathbf{v}_i)_{i\geq -2}$ has dimension 3, then the sequence $(\mathrm{w}_k)_{k\geq 0}$ is admissible.

The last part of the proposition is implied by Lemma 3.5. The first part comes from Proposition 3.8 below.

Proposition 3.7. *Let $(\mathbf{v}_i)_{i\geq -2}$ be as in Proposition 3.6, and for each $k \geq 0$, set $\mathrm{w}_k := \mathbf{v}_{t_k+1}\mathbf{v}_{t_k}^{-1}$. Then, we have the following properties:*

(i) $\mathbf{v}_{t_k+\ell} = \mathrm{w}_k^{\ell}\mathbf{v}_{t_k} = \mathrm{w}_k^{\ell+1}\mathbf{v}_{\psi(t_k)}$ *for $k \geq 1$ and $0 \leq \ell \leq s_{k+1}$.*

(ii) $\mathbf{v}_{j+1} = \mathrm{w}_k\mathbf{v}_j$ *for $k \geq 0$ and $t_k \leq j < t_{k+1}$.*

(iii) $\mathbf{v}_j = \mathrm{w}_k\mathbf{v}_{\psi(j)}$ *for $k \geq 1$ and $t_k \leq j < t_{k+1}$.*

Moreover, the sequence $(\mathrm{w}_k)_{k\geq 0}$ is a ψ-Sturmian sequence in $\mathrm{GL}_2(\mathbb{R})$.

Proof. Since $(t_0, t_1) = (-1, 0)$, the case $k = 0$ of (ii) is trivial by definition of w_0. Let k, j be integers with $k \geq 1$ and $t_k \leq j < t_{k+1}$. Recall that $\psi(j) = j - 1$ for each j with $t_k < j < t_{k+1}$, so that, using successively (19), we find

$$\mathbf{v}_{j+1}\mathbf{v}_j^{-1} = \mathbf{v}_j\mathbf{v}_{\psi(j)}^{-1} = \cdots = \mathbf{v}_{t_k+1}\mathbf{v}_{t_k}^{-1} = \mathrm{w}_k = \mathbf{v}_{t_k}\mathbf{v}_{\psi(t_k)}^{-1},$$

which proves (ii) and (iii). Assertion (i) is a consequence of (ii) and (iii).
Finally, by (19), we have $\mathrm{w}_{k+1} = \mathbf{v}_{t_{k+1}}\mathbf{v}_{\psi(t_{k+1})}^{-1} = (\mathbf{v}_{t_k+s_{k+1}}\mathbf{v}_{t_k}^{-1})(\mathbf{v}_{t_k}\mathbf{v}_{t_k-1}^{-1})$. Using (i) and (ii), we find $\mathrm{w}_{k+1} = \mathrm{w}_k^{s_{k+1}}\mathrm{w}_{k-1}$, hence the last part of the proposition. □

Proposition 3.8. *Let $(\mathbf{v}_i)_{i\geq -2}$ be as in Proposition 3.6, and define the ψ-Sturmian sequence $(\mathrm{w}_k)_{k\geq 0}$ as in Proposition 3.7. Setting $N := {}^t(\mathbf{v}_{-1}\mathbf{v}_0^{-1}\mathbf{v}_{-2})$, we have $\mathbf{v}_{-2} = \mathrm{w}_0{}^tN$ and*

$$\mathbf{v}_{t_k+\ell} = \mathrm{w}_k^{\ell+1}\mathrm{w}_{k-1}N_k \quad (k \geq 1 \text{ and } 0 \leq \ell < s_{k+1}), \tag{20}$$

where $N_k = N$ if k is even, $N_k = {}^tN$ else.

Proof. For each $k \geq 0$, set

$$N'_{k+1} := \mathrm{w}_k^{-1}\mathbf{v}_{\psi(t_{k+1})} = \mathbf{v}_{t_k}\mathbf{v}_{t_k+1}^{-1}\mathbf{v}_{t_k-1}. \tag{21}$$

By (19), we have

$$\mathbf{v}_{t_{k+1}}\mathbf{v}_{t_{k+1}+1}^{-1} = \mathbf{v}_{\psi(t_{k+1})}\mathbf{v}_{t_{k+1}}^{-1} \quad \text{and} \quad \mathbf{v}_{t_{k+1}}^{-1}\mathbf{v}_{t_{k+1}-1} = \cdots = \mathbf{v}_{t_k+1}^{-1}\mathbf{v}_{t_k},$$

from which we deduce

$$N'_{k+2} = \mathbf{v}_{t_{k+1}}\mathbf{v}_{t_{k+1}+1}^{-1}\mathbf{v}_{t_{k+1}-1} = \mathbf{v}_{\psi(t_{k+1})}\mathbf{v}_{t_{k+1}}^{-1}\mathbf{v}_{t_{k+1}-1} = \mathbf{v}_{t_k-1}\mathbf{v}_{t_k+1}^{-1}\mathbf{v}_{t_k} = {}^tN'_{k+1}.$$

Since ${}^tN = N_1 = N'_1$, it implies that $N_k = N'_k$ for each $k \geq 1$, and (21) provide the identity $\mathbf{v}_{\psi(t_{k+1})} = \mathrm{w}_kN_{k+1}$. With $k = 0$, this gives $\mathbf{v}_{-2} = \mathrm{w}_0{}^tN$. More generally, combined with Proposition 3.7, this yields (20). □

3.3. **New key-identities.**

Recall that $\mathbb{R}^3$ is identified to the space of symmetric matrices of $\mathrm{Mat}_{2\times 2}(\mathbb{R})$, so that $X \wedge Y$ is well defined for any symmetric matrices $X, Y \in \mathrm{Mat}_{2\times 2}(\mathbb{R})$. We denote by J the matrix defined as in (10). The goal of this section is to give another expression for the sequence $(\mathbf{z}_i)_{i\geq -1}$ of Definition 3.4. This will allow us to compute the exponents ω_2^* and $\widehat{\omega}_2^*$ of a Sturmian number in Section 5.3. See the introduction and (4) for the motivation of the following definition.

Definition 3.9. We define the morphism $\mathrm{U} : \mathrm{Mat}_{2\times 2}(\mathbb{R}) \to \mathrm{Mat}_{2\times 2}(\mathbb{R})$ by

$$\mathrm{U}\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right) := \begin{pmatrix} -c & a-d \\ a-d & b \end{pmatrix}.$$

Note that for each $X \in \mathrm{Mat}_{2\times 2}(\mathbb{R})$, the matrix $\mathrm{U}(X)$ is symmetric, $U(\mathrm{Adj}(X)) = -U(X)$, and $\mathrm{U}(X) = 0$ if and only if X is proportional to Id.

Definition 3.10. Let $(\mathrm{w}_k)_{k\geq 0}$ be a ψ-Sturmian sequence in $\mathrm{GL}_2(\mathbb{R})$ and set $\mathrm{w}_{-1} := \mathrm{w}_0^{-1}\mathrm{w}_1$. We associate to $(\mathrm{w}_k)_{k\geq 0}$ two sequences $(\mathbf{a}_i)_{i\geq -1}$ and $(\mathbf{b}_i)_{i\geq -1}$ of symmetric matrices as follows. For $k, \ell \in \mathbb{N}$ with $k \geq 0$ and $0 \leq \ell < s_{k+1}$, we define $\mathbf{b}_{t_k+\ell} = U(\mathrm{w}_k^\ell \mathrm{w}_{k-1})$ and

$$\mathbf{a}_{t_k+\ell} = (-1)^{k+1}\mathbf{b}_{t_{k+1}} \wedge \mathbf{b}_{t_k+\ell} = (-1)^{k+1} U(\mathrm{w}_k) \wedge U(\mathrm{w}_k^\ell \mathrm{w}_{k-1}).$$

The main result of this section is the following.

Proposition 3.11. *Let $(\mathrm{w}_k)_{k\geq 0}$ be an admissible ψ-Sturmian sequence in $\mathrm{GL}_2(\mathbb{R})$. Then*

$$\mathbf{a}_i = \mathbf{y}_i \quad \text{and} \quad \mathbf{b}_i = \det(N)^{-1}\mathbf{z}_i$$

for each $i \geq -1$, where $(\mathbf{y}_i)_{i\geq -2}$ and $(\mathbf{z}_i)_{i\geq -1}$ are the sequences of symmetric matrices given by Definition 3.4 with $N = (\mathrm{Id} - \mathrm{w}_1^{-1}\mathrm{w}_0^{-1}\mathrm{w}_1\mathrm{w}_0)J$.

Before proving this result, let us state some elementary identities satisfied by U. They can easily be obtained by a direct computation, details are left to the reader.

Proposition 3.12. *For each $X, Y \in \mathrm{Mat}_{2\times 2}(\mathbb{R})$, we have the following properties.*

(i) $\mathrm{U}(X) \wedge \mathrm{U}(Y) = 0$ *if and only if $XY = YX$. More precisely*

$$\mathrm{U}(X) \wedge \mathrm{U}(Y) = -(XY - YX)J. \tag{22}$$

(ii) *If X, Y are invertible, then* $\mathrm{U}(X) \wedge \mathrm{U}(Y) = -XY(\mathrm{Id} - Y^{-1}X^{-1}YX)J$.

(iii) *If X, Y are symmetric, then* $U(X\mathrm{Adj}(Y)) = -X \wedge Y$.

It is also interesting to notice the following identities (although we will not need them in this paper)

$$U(XY) + U(YX) = \mathrm{Tr}(X)U(Y) + \mathrm{Tr}(Y)U(X),$$

and

$$\mathrm{U}(PXP^{-1}) \wedge \mathrm{U}(PYP^{-1}) = \det(P)^{-1}P(\mathrm{U}(X) \wedge \mathrm{U}(Y))^tP,$$

valid for each $X, Y \in \mathrm{Mat}_{2\times 2}(\mathbb{R})$ and $P \in \mathrm{GL}_2(\mathbb{R})$. We get the first one by a direct computation, and the last one is a consequence of (22) and the equality $\mathrm{Adj}(P)J = J({}^tP)$.

Proof of Proposition 3.11. Let k, ℓ with $k \geq 0$ and $0 \leq \ell < s_{k+1}$. We first prove the formula $\mathbf{b}_i = \det(N)^{-1}\mathbf{z}_i$ for each $i \geq -1$. We can derive from (11) the general identity $\mathrm{w}_k\mathrm{w}_{k-1}N_k = \mathrm{w}_{k-1}\mathrm{w}_k N_{k+1}$ (see [24, Proposition 3.4]). Combined with (15), we obtain $\mathbf{y}_{t_k+\ell} = \mathrm{w}_k^\ell \mathrm{w}_{k-1}\mathrm{w}_k N_{k+1}$. On the other hand Eq. (16) gives $\mathbf{y}_{\psi(t_{k+1})} = \mathrm{w}_k N_{k+1}$, so that

$$\mathbf{y}_{t_k+\ell}\mathbf{y}_{\psi(t_{k+1})}^{-1} = \mathrm{w}_k^\ell \mathrm{w}_{k-1}.$$

Since $\mathbf{y}_{t_k+\ell}$ and $\mathbf{y}_{\psi(t_{k+1})}$ are symmetric, together with assertion (iii) of Proposition 3.12 and (15), this yields

$$\mathbf{b}_{t_k+\ell} = U(\mathrm{w}_k^\ell \mathrm{w}_{k-1}) = \det(\mathbf{y}_{\psi(t_{k+1})})^{-1}\mathbf{y}_{\psi(t_{k+1})} \wedge \mathbf{y}_{t_k+\ell} = \det(N)^{-1}\mathbf{z}_{t_k+\ell}.$$

Now we prove that $\mathbf{a}_{t_k+\ell} = \mathbf{y}_{t_k+\ell}$. Assertion (ii) of Proposition 3.12 gives

$$\mathbf{a}_{t_k+\ell} = (-1)^{k+1}U(\mathrm{w}_k) \wedge U(\mathrm{w}_k^\ell \mathrm{w}_{k-1}) = (-1)^k \mathrm{w}_k^{\ell+1}\mathrm{w}_{k-1}(\mathrm{Id} - \mathrm{w}_{k-1}^{-1}\mathrm{w}_k^{-1}\mathrm{w}_{k-1}\mathrm{w}_k)J.$$

We conclude by noticing that $\mathrm{w}_{k+1}^{-1}\mathrm{w}_k^{-1}\mathrm{w}_{k+1}\mathrm{w}_k = \mathrm{w}_{k-1}^{-1}\mathrm{w}_k^{-1}\mathrm{w}_{k-1}\mathrm{w}_k$ combined with (14) implies

$$(-1)^k(\mathrm{Id} - \mathrm{w}_{k-1}^{-1}\mathrm{w}_k^{-1}\mathrm{w}_{k-1}\mathrm{w}_k)J = N_k.$$

□

4. Estimates for Sturmian sequences of matrices

We keep the notation of Section 3 for $\mathbf{s}$ and $\psi = \psi_{\mathbf{s}}$. In §4.1, we establish a new simple criterion so that a given admissible ψ-Sturmian sequence $(\mathrm{w}_k)_{k\geq 0}$ has multiplicative growth. In §4.2, we solve the delicate question (and essential for our study) of knowing how to control the content of w_k, assuming that $(\mathrm{w}_k)_{k\geq 0}$ is defined over $\mathbb{Q}$. Altogether with the results of the previous section, we finally establish a new characterization of *Sturm*($\mathbf{s}$) in §4.3.

The next result will allow us to eliminate the degenerate situation where a ψ-Sturmian sequence is admissible with an antisymmetric matrix N.

Lemma 4.1. *Let $(\mathrm{w}_k)_{k\geq 0}$ be a ψ-Sturmian sequence in $\mathrm{GL}_2(\mathbb{R})$ and $N \in \mathrm{GL}_2(\mathbb{R})$ be such that $\mathrm{w}_0{}^tN$, $\mathrm{w}_1 N$ and $\mathrm{w}_1\mathrm{w}_0{}^tN$ are symmetric. Suppose that the sequence $(\mathbf{y}_i)_{i\geq -2}$ of symmetric matrices associated to $(\mathrm{w}_k)_{k\geq 0}$ and N as in Definition 3.4 converges projectively. Then N is not antisymmetric.*

Proof. By contradiction, suppose that N is antisymmetric, and write $N = \alpha J$ with $\alpha \in \mathbb{R}\setminus\{0\}$. Then, by (12), we have $\mathrm{w}_0\mathrm{w}_1 = -\mathrm{w}_1\mathrm{w}_0 \neq \mathrm{w}_1\mathrm{w}_0$. We claim that w_0^ℓ (resp. w_1^ℓ) is proportional to Id if ℓ is even, and w_0 (resp. w_1) if ℓ is odd. Indeed, since $J^{-1} = -J = {}^tJ$, we have $\alpha\mathrm{w}_0 = \mathbf{y}_{-2}J$ and $\alpha\mathrm{w}_1 = -\mathbf{y}_{-1}J$, and we conclude with the identity $\mathbf{x}J\mathbf{x}J = -\det(\mathbf{x})\mathrm{Id}$ valid for each symmetric matrix $\mathbf{x} \in \mathrm{GL}_2(\mathbb{R})$. As a consequence, for any $i \geq -2$, the non-zero symmetric matrix $\mathbf{y}_i$ is proportional to either w_0J, w_1J or $\mathrm{w}_1\mathrm{w}_0J$. Since by Lemma 3.5 the points $\mathbf{y}_{t_k-1}$, $\mathbf{y}_{t_k}$, $\mathbf{y}_{t_k+1}$ are linearly independent for each $k \geq 0$, we deduce that projectively, the sequence $(\mathbf{y}_i)_{i\geq -2}$ has exactly three accumulation points, a contradiction. □

4.1. **Multiplicative growth property.**

Showing the multiplicative growth of an admissible ψ-Sturmian sequence in $\mathrm{GL}_2(\mathbb{R})$ is difficult, partly because of the lack of control of the signs of the coefficients: opposite terms can cancel out. The proof of Lemma 5.1 of [30] gives a useful criterion for showing the multiplicative growth if w_0 and w_1 are of a certain type. The examples given by Roy in [30] (see also [24, Section 8.1]) satisfy this criterion and allow us to avoid the alluded difficulty (see also Example 2 of [29] for an example of construction of extremal numbers which does not satisfies the criterion of [30, Lemma 5.1]). We establish a new condition under which an admissible ψ-Sturmian sequence has multiplicative growth. Recall that the matrix J is defined by (10).

Proposition 4.2. *Let $(\mathrm{w}_k)_{k\geq 0}$ be an admissible ψ-Sturmian sequence in $\mathrm{GL}_2(\mathbb{Q})$ and let $(\mathbf{y}_i)_{i\geq -2}$ be the sequence of symmetric matrices associated to $(\mathrm{w}_k)_{k\geq 0}$ by Definition 3.4. If $(\mathbf{y}_i)_{i\geq -2}$ converges projectively to a point $\mathbf{y} = (1, \xi, \xi^2)$, where $\xi \in \mathbb{R}$ is neither rational nor quadratic, then $(\mathrm{w}_k)_{k\geq 0}$ has multiplicative growth.*

Proposition 4.2 is a corollary of Proposition 4.4 below.

Lemma 4.3. *Let ξ be a real number neither rational nor quadratic. Let N be a positive integer and $B_1, \dots, B_N \in \mathrm{Mat}_{2\times 2}(\mathbb{Q}) \setminus (\mathbb{Q}J)$. Then $\Xi B_N \Xi B_{N-1} \dots \Xi B_1 \neq 0$, where Ξ is the symmetric matrix corresponding to $(1, \xi, \xi^2)$.*

Proof. Since the image of Ξ is equal to $\langle \mathbf{x} \rangle$, where $\mathbf{x} = {}^t(1, \xi)$, it suffices to prove that for any $B \in \mathrm{Mat}_{2\times 2}(\mathbb{Q}) \setminus (\mathbb{Q}J)$, the vector $\Xi B \mathbf{x} \in \langle \mathbf{x} \rangle$ is non-zero. Let $B \in \mathrm{Mat}_{2\times 2}(\mathbb{Q})$ and write

$$B = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

Then $\Xi B \mathbf{x} = 0$ if and only if $a + (b + c)\xi + d\xi^2 = 0$. Since ξ is neither rational nor quadratic, it is equivalent to $B \in \mathbb{Q}J$. □

Proposition 4.4. *Let $(\mathrm{w}_k)_{k\geq 0}$ and $(\mathbf{y}_i)_{i\geq -2}$ satisfying the hypotheses of Proposition 4.2. Let N be a positive integer and $\mathcal{B}$ be a finite subset of $\mathrm{Mat}_{2\times 2}(\mathbb{Q}) \setminus (\mathbb{Q}J)$. Then, there are a constant $c > 0$ and an index i_0 which only depend on $\mathcal{B}$, N and ξ, such that, for any indices $j_1, \dots, j_N \geq i_0$ and any matrices $B_1, \dots, B_N \in \mathcal{B}$, we have*

$$c^{-1} \prod_{k=1}^{N} \|\mathbf{y}_{j_k}\| \leq \|\mathbf{y}_{j_1} B_1 \cdots \mathbf{y}_{j_N} B_N\| \leq c \prod_{k=1}^{N} \|\mathbf{y}_{j_k}\|. \tag{23}$$

Furthermore, the sequence $(\mathrm{w}_k)_{k\geq 0}$ has multiplicative growth.

Proof. Since $\mathcal{B}$ is finite, the set of N-tuples of matrices $(B_1, \dots, B_N) \in \mathcal{B}^N$ is finite. We denote by Ξ the symmetric matrix corresponding to $(1, \xi, \xi^2)$. By Lemma 4.3, there is a constant $c_1 > 0$ (which depends only on $\mathcal{B}$, N and ξ), such that

$$\frac{1}{c_1} \leq \|\Xi B_1 \cdots \Xi B_N\| \leq c_1 \tag{24}$$

for each $(B_1, \dots, B_N) \in \mathcal{B}^N$. Write $\mathbf{y}_i = \begin{pmatrix} y_{i,0} & y_{i,1} \\ y_{i,1} & y_{i,2} \end{pmatrix}$ for each $i \geq -2$. By hypothesis $y_{i,0}^{-1}\mathbf{y}_i$ tends to Ξ, in particular $|y_{i_0}| \asymp \|\mathbf{y}_i\|$. Fix $(B_1, \dots, B_N) \in \mathcal{B}$. From the above the matrix product

$$\Big(\prod_{k=1}^{N} y_{j_k,0}\Big)^{-1} \mathbf{y}_{j_1} B_1 \cdots \mathbf{y}_{j_N} B_N$$

tends to $\Xi B_1 \cdots \Xi B_N$ as i tends to infinity, uniformly in $j_1, \dots, j_N \geq i$. Thus, by (24), there exist a constant $c > 0$ and an index i_0 such that Eq. (23) holds for each $j_1, \dots, j_N \geq i_0$. Since $\mathcal{B}^N$ is finite, we may suppose that this estimate is satisfied for all $(B_1, \dots, B_N) \in \mathcal{B}^N$, which ends the proof of (23). We now prove that $(\mathrm{w}_k)_{k\geq 0}$ has multiplicative growth. Fix k, ℓ with $k \geq 1$ and $0 \leq \ell \leq s_{k+1}$. By (16), we have $\mathrm{w}_k = \mathbf{y}_{\psi(t_{k+1})} N_{k+1}^{-1}$ and $\mathrm{w}_k^{\ell}\mathrm{w}_{k-1} = \mathbf{y}_{t_k+\ell-1} N_k^{-1}$ if $\ell > 0$. In particular $\|\mathrm{w}_k\| \asymp \|\mathbf{y}_{\psi(t_{k+1})}\|$ and $\|\mathrm{w}_k^{\ell}\mathrm{w}_{k-1}\| \asymp \|\mathbf{y}_{t_k+\ell-1}\|$ if $\ell > 0$. Moreover, the matrices N^{-1} and ${}^tN^{-1}$ are not proportional to J according to Lemma 4.1. By writing

$$\mathrm{w}_k^{\ell+1}\mathrm{w}_{k-1} = \mathrm{w}_k(\mathrm{w}_k^{\ell}\mathrm{w}_{k-1}) = \begin{cases} \mathbf{y}_{\psi(t_{k+1})} N_{k+1}^{-1} \mathbf{y}_{t_k+\ell-1} N_k^{-1} & \text{if } \ell > 0, \\ \mathbf{y}_{\psi(t_{k+1})} N_{k+1}^{-1} \mathbf{y}_{\psi(t_k)} N_k^{-1} & \text{if } \ell = 0, \end{cases}$$

and by using (23), we conclude easily that $\|\mathrm{w}_k^{\ell+1}\mathrm{w}_{k-1}\| \asymp \|\mathrm{w}_k\| \|\mathrm{w}_k^{\ell}\mathrm{w}_{k-1}\|$. □

We can deduce from the proofs of [24, Proposition 6.1 and Proposition 6.5] the following result (note that in [24] we suppose that $\mathrm{w}_k \in \mathrm{Mat}_{2\times 2}(\mathbb{Z})$, but this hypothesis is not needed to get the estimates of our proposition).

Proposition 4.5. *Let $(\mathrm{w}_k)_{k\geq 0}$ be an unbounded admissible ψ-Sturmian sequence in $\mathrm{GL}_2(\mathbb{R})$ with multiplicative growth, and denote by $(\mathbf{y}_i)_{i\geq -2}$ and $(\mathbf{z}_i)_{i\geq -1}$ the associated sequences of symmetric* *matrices (see Definition 3.4). Suppose that there exists $\delta < 2$ such that $|\det(\mathrm{w}_k)| \ll \|\mathrm{w}_k\|^\delta$ for each $k \geq 0$. Then, the sequence $(\mathbf{y}_i)_{i\geq -2}$ converges projectively to a point $\Xi = (1, \xi, \xi^2)$, and*

$$\|\mathbf{y}_i \wedge \Xi\| \asymp \frac{|\det(\mathbf{y}_i)|}{\|\mathbf{y}_i\|}, \quad \|\mathbf{z}_i\| \asymp \|\mathbf{y}_{\psi(i)}\| \quad \textit{and} \quad |\mathbf{z}_i \cdot \Xi| \asymp \frac{|\det(\mathbf{y}_i)|}{\|\mathbf{y}_{i+1}\|}.$$

4.2. **Estimates for the norms and the contents.**

Proposition 4.6 below generalizes, among others, the first part of Proposition 5.6 of [24]. The growth of the contents (27) was originally proven in [23, Chapter 3] in a different way. This is one of the most delicate points. We are grateful to Damien Roy for pointing us out a much shorter proof than the original one. Recall that the sequence $\mathbf{s}$ is not necessarily bounded and that U denotes the map introduced in Section 3.3. We define the sequence $(p_k)_{k\geq -1}$ by

$$(p_{-1}, p_0) = (0, 1) \quad \text{and} \quad p_{k+1} = s_{k+1}p_k + p_{k-1} \quad (k \geq 0). \tag{25}$$

Proposition 4.6. *Let $(\mathrm{w}_k)_{k\geq 0}$ be a ψ-Sturmian sequence in $\mathrm{GL}_2(\mathbb{R})$ with multiplicative growth such that $(\|\mathrm{w}_k\|)_{k\geq 0}$ is unbounded. Then, there exist real numbers α, β, with $\beta > 0$, such that*

$$|\det(\mathrm{w}_k)| \asymp e^{\alpha p_k} \quad \textit{and} \quad \|\mathrm{w}_k\| \asymp e^{\beta p_k} =: W_k, \tag{26}$$

as $k \geq 0$ tends to infinity. If w_k is defined over $\mathbb{Q}$ for each $k \geq 0$, then there is $\varrho \in \mathbb{R}$ such that

$$\mathrm{cont}(\mathrm{w}_k) = e^{p_k(\varrho + o(1))}, \tag{27}$$

as k tends to infinity. Suppose furthermore $(\mathrm{w}_k)_{k\geq 0}$ admissible, and that either $\mathbf{s}$ is bounded or $\alpha = 2\varrho$. Given $i = t_k + \ell$ with $k \geq 1$ and $0 \leq \ell < s_{k+1}$, we define

$$Y_i := W_k^{\ell+1} W_{k-1} \quad \textit{and} \quad Z_i = W_k^\ell W_{k-1}.$$

Then, as i tends to infinity, we have $\|\mathrm{w}_k^{\ell+1}\mathrm{w}_{k-1}\| = Y_i^{1+o(1)}$, $\|\mathrm{w}_k^\ell \mathrm{w}_{k-1}\| = Z_i^{1+o(1)}$, as well as

$$\mathrm{cont}(\mathrm{w}_k^{\ell+1}\mathrm{w}_{k-1}) = Y_i^{\varrho/\beta + o(1)} \quad \textit{and} \quad \mathrm{cont}(\mathrm{U}(\mathrm{w}_k^\ell \mathrm{w}_{k-1})) = Z_i^{\varrho/\beta + o(1)}. \tag{28}$$

The proof of this result is at the end of this section. With that goal in mind, let us introduce for each $i \in \mathbb{N}$ the sequence $(q_k^{(i)})_{k\in\mathbb{Z}}$, defined by

$$q_k^{(i)} = \begin{cases} 0 & \text{if } k < i \\ 1 & \text{if } k = i \\ s_k q_{k-1}^{(i)} + q_{k-2}^{(i)} & \text{if } k > i. \end{cases}$$

Note that $(q_k^{(0)})_{k\geq -1}$ is the sequence $(p_k)_{k\geq -1}$ of (25). Moreover, the theory of continued fractions (see for instance [35, Chapter I]) ensures that for each $k \geq i > 0$ we have

$$\frac{q_k^{(i-1)}}{q_k^{(i)}} = [s_i; s_{i+1}, \ldots, s_k] \quad \text{and} \quad |q_k^{(i-1)} - \xi_i q_k^{(i)}| \leq \frac{1}{q_{k+1}^{(i)}}, \tag{29}$$

where $\xi_i := [s_i; s_{i+1}, \ldots]$. Note that the right-hand side of (29) still holds (and is an equality) for $k = i - 1$. Furthermore, $\xi_i \geq s_i \geq 1$ and $\xi_i = s_i + 1/\xi_{i+1}$, hence

$$\xi_i \xi_{i+1} = s_i \xi_{i+1} + 1 \geq 2 \quad (i \geq 1). \tag{30}$$

Lemma 4.7. *For each $i \geq 0$ the quotient $q_k^{(i)}/q_k^{(0)}$ tends to $a_i := 1/(\xi_1 \ldots \xi_i)$ as k tends to infinity, where $a_0 = 1$. Moreover, $\sum_{i\geq 0} s_i a_i < \infty$ and there exist $A, B > 0$ with the following properties.*

(i) $\displaystyle\sum_{i\geq 0} s_i |q_k^{(i)} - a_i q_k^{(0)}| \leq A$ *for each $k \geq 0$;*

(ii) $q_k^{(i)}/q_k^{(0)} \leq B a_i$ *for each $k, i \geq 0$.*

Proof. Let $i \geq 0$. Then $q_k^{(i)}/q_k^{(0)} = a_i + o(1)$ as k tends to infinity, since for each $j \geq 1$ the quotient $q_k^{(j-1)}/q_k^{(j)}$ tends to ξ_j. For each $a, b \in \mathbb{R}$, we set $\delta_{a\leq b} = 1$ if $a \leq b$, and 0 else. Given $k \geq 0$, we have

$$|a_i q_k^{(0)} - q_k^{(i)}| = \Big|\sum_{j=1}^{i} \frac{1}{\xi_j \cdots \xi_i}(q_k^{(j-1)} - \xi_j q_k^{(j)})\Big| \leq \sum_{j\geq 1} \frac{\delta_{j\leq i}}{\xi_j \cdots \xi_i} \frac{\delta_{j\leq k+1}}{q_{k+1}^{(j)}},$$

where the last estimate is obtained by noticing that the term in the first sum vanishes if $k < j-1$, and by using the upper bound given by (29) for the indices j with $j \leq k+1$. This yields

$$\sum_{i\geq 0} s_i |a_i q_k^{(0)} - q_k^{(i)}| \leq \sum_{i\geq 0}\sum_{j\geq 1} s_i \frac{\delta_{j\leq i}}{\xi_j \cdots \xi_i} \frac{\delta_{j\leq k+1}}{q_{k+1}^{(j)}} = \sum_{j=1}^{k+1} \frac{1}{q_{k+1}^{(j)}} \sum_{i\geq j} \frac{s_i}{\xi_j \cdots \xi_i}. \tag{31}$$

On the one hand, using the upper bound $s_i \leq \xi_i$ and (30), we get

$$\sum_{i\geq j} \frac{s_i}{\xi_j \cdots \xi_i} \leq \sum_{i\geq j} \frac{1}{\xi_j \cdots \xi_{i-1}} \leq \sum_{i\geq j} \frac{1}{2^{\lfloor (i-j)/2\rfloor}} = 2\sum_{k\geq 0}\frac{1}{2^k} = 1.$$

Taking $j = 1$ in the above, we get $\sum_{i\geq 1} s_i a_i \leq 1$. On the other hand, since the golden ratio $\gamma = (1+\sqrt{5})/2$ satisfies $\gamma^j = \gamma^{j-1} + \gamma^{j-2}$ for each $j \geq 0$, it is easy to check by induction that $q_{k+1}^{(j)} \geq \gamma^{k-j}$ for each $j \leq k+1$. Together with (31), we find

$$\sum_{i\geq 0} s_i |a_i q_k^{(0)} - q_k^{(i)}| \leq \sum_{j=1}^{k+1} \frac{1}{\gamma^{k-j}} \leq \sum_{j\geq 0} \frac{1}{\gamma^{j-1}} =: A < +\infty,$$

hence (i). We now prove (ii). We may assume that $k \geq i$ since $q_k^{(i)} = 0$ if $k < i$. If $k > i$, then we have $[s_i; s_{i+1}, \dots, s_k] \geq \xi_i(1-\gamma^{i-k})$, since and $\xi_i \geq 1$ and $|[s_i; s_{i+1}, \dots, s_k] - \xi_i| \leq 1/q_{k+1}^{(i)} \leq \gamma^{i-k}$ by (29). If $k = i$, we simply have $[s_i; \dots, s_k] = s_i \geq \xi_i/2$. Combining these lower bounds, we find

$$\frac{q_k^{(i)}}{q_k^{(0)}} = \frac{1}{[s_1; s_2, \dots, s_k] \cdots [s_i; s_{i+1}, \dots, s_k]} \leq \frac{B}{\xi_1 \cdots \xi_i}, \quad \text{where } B := \frac{2}{\prod_{j\geq 1}(1-\gamma^{-j})} < \infty.$$

□

Lemma 4.8. *Let $(r_k)_{k\geq 0}$ be a sequence of real numbers. Set $\varepsilon_k := r_k - s_k r_{k-1} - r_{k-2}$ for each $k \geq 0$, with $(r_{-2}, r_{-1}) = (0,0)$. Then*

$$r_k = \sum_{i=0}^{k} \varepsilon_i q_k^{(i)} \qquad (k \geq -2), \tag{32}$$

with the convention that the sum on the right-hand side is equal to 0 if $k < 0$.

Proof. We prove Eq. (32) by induction on $k \geq -2$. It trivially holds for $k = -2$ and $k = -1$. Suppose now that (32) is satisfied for $-2, -1, \dots, k-1$, where k is a given integer ≥ 0. Then, since $q_k^{(k)} = 1$ and $q_{k-2}^{(k-1)} = 0$, we obtain

$$r_k - \varepsilon_k q_k^{(k)} = s_k r_{k-1} + r_{k-2} = \sum_{i=0}^{k-1} \varepsilon_i (s_k q_{k-1}^{(i)} + q_{k-2}^{(i)}) = \sum_{i=0}^{k-1} \varepsilon_i q_k^{(i)}.$$

□

Lemma 4.9. *Let $(r_k)_{k\geq 0}$ and $(\varepsilon_k)_{k\geq 0}$ be as in Lemma 4.8, and suppose that $\varepsilon_k = \mathcal{O}(s_k)$. Then, there exists $\lambda \in \mathbb{R}$ such that*

$$r_k = \lambda p_k + \mathcal{O}(1) \qquad (k \geq 0). \tag{33}$$

Remark. Eq. (33) can be deduced from the proof of [24, Proposition 5.5].

Proof. Set $\lambda := \sum_{i\geq 0}\varepsilon_i a_i$ and let $c>0$ be such that $|\varepsilon_i|\leq cs_i$ for each $i\geq 0$. We have $\lambda\in\mathbb{R}$ by Lemma 4.7. Then, using (32) and (i) of Lemma 4.7, we obtain for each $k\geq 0$

$$|r_k-\lambda q_k^{(0)}| = |\sum_{i\geq 0}\varepsilon_i(q_k^{(i)}-a_iq_k^{(0)})| \leq c\sum_{i\geq 0}s_i|q_k^{(i)}-a_iq_k^{(0)}| \leq cA.$$

□

Surprisingly, under the weaker assumption $\varepsilon_k\geq 0$, the quotient r_k/p_k still converges, as soon as it is bounded (see below).

Lemma 4.10. *Let $(r_k)_{k\geq 0}$ and $(\varepsilon_k)_{k\geq 0}$ be as in Lemma 4.8, and suppose that for any sufficiently large k, we have $\varepsilon_k\geq 0$ and $r_k\leq Mp_k$ for a constant $M>0$ independent of k. Then r_k/p_k has a limit as k tends to infinity.*

Proof. Let k,ℓ be integers with $k\geq\ell\geq 0$. By (32), if ℓ is large enough, then we have $\varepsilon_i\geq 0$ for each $i\geq\ell$, and

$$\sum_{i=0}^{\ell}\varepsilon_i\frac{q_k^{(i)}}{q_k^{(0)}} \leq \sum_{i=0}^{k}\varepsilon_i\frac{q_k^{(i)}}{q_k^{(0)}} = \frac{r_k}{q_k^{(0)}} \leq M.$$

Yet, the quotient $q_k^{(i)}/q_k^{(0)}$ tends to a_i. By letting first k, then ℓ, tend to infinity in the left-hand side, this shows that the series $\sum_{i\geq 0}\varepsilon_i a_i$ converges absolutely. So, we can apply the dominated convergence Theorem by (ii) of Lemma 4.7. Defining $\delta_{i\leq k}=1$ if $i\leq k$, and 0 otherwise, we get

$$\lim_{k\to\infty}\frac{r_k}{q_k^{(0)}} = \lim_{k\to\infty}\sum_{i\geq 0}\delta_{i\leq k}\varepsilon_i\frac{q_k^{(i)}}{q_k^{(0)}} = \sum_{i\geq 0}\lim_{k\to\infty}\delta_{i\leq k}\varepsilon_i\frac{q_k^{(i)}}{q_k^{(0)}} = \sum_{i\geq 0}\varepsilon_i a_i\in\mathbb{R}.$$

□

Proof of Proposition 4.6. Let $(\mathrm{w}_k)_{k\geq 0}$ be a ψ-Sturmian sequence as in Proposition 4.6. For each $k\geq 2$, we have $\mathrm{w}_k=\mathrm{w}_{k-1}^{s_k}\mathrm{w}_{k-2}$. Consequently, the sequence $(r_k)_{k\geq 0}:=(\log|\det(\mathrm{w}_k)|)_{k\geq 0}$ satisfies $r_k-s_kr_{k-1}-r_{k-2}=0$ for each $k\geq 2$, and we obtain the estimate for $|\det(\mathrm{w}_k)|$ in (26) by applying Lemma 4.9. Similarly, the multiplicative growth gives

$$\log\|\mathrm{w}_k\|-s_k\log\|\mathrm{w}_{k-1}\|-\log\|\mathrm{w}_{k-1}\| = \mathcal{O}(s_k),$$

and therefore, we can also apply Lemma 4.9 with the sequence $(r_k)_{k\geq 0}:=(\log\|\mathrm{w}_k\|)_{k\geq 0}$. Hence the estimate for $\|\mathrm{w}_k\|$ in (26). Note that since $(\|\mathrm{w}_k\|)_{k\geq 0}$ is unbounded, we must have $\beta>0$.

Now, suppose that $(\mathrm{w}_k)_{k\geq 0}$ is defined over $\mathbb{Q}$ and consider $(r_k)_{k\geq 0}:=(\log\mathrm{cont}(\mathrm{w}_k))_{k\geq 0}$. Given $k\geq 2$, the identity $\mathrm{w}_k=\mathrm{w}_{k-1}^{s_k}\mathrm{w}_{k-2}$ implies that $r_k\geq s_kr_{k-1}+r_{k-2}$ by definition of the content. Moreover, Eq. (26) yields $r_k\leq\log\|\mathrm{w}_k\|\ll\beta p_k$ as k tends to infinity. Lemma 4.10 gives (27).

Suppose in addition that $(\mathrm{w}_k)_{k\geq 0}$ is admissible, and that $\mathbf{s}$ is bounded or $\alpha=2\varrho$. Let $i=t_k+\ell$ with $k\geq 1$ and $0\leq\ell<s_{k+1}$. The estimates $\|\mathrm{w}_k^{\ell+1}\mathrm{w}_{k-1}\|=Y_i^{1+o(1)}$, $\|\mathrm{w}_k^{\ell}\mathrm{w}_{k-1}\|=Z_i^{1+o(1)}$ are obtained by multiplicative growth (and since $\|\mathrm{w}_k\|\asymp W_k$ tends to infinity). Furthermore

$$\mathrm{cont}(\mathrm{U}(\mathrm{w}_k))\mathrm{cont}(\mathrm{U}(\mathrm{w}_k^{\ell}\mathrm{w}_{k-1})) \leq \mathrm{cont}(\mathrm{U}(\mathrm{w}_k)\wedge\mathrm{U}(\mathrm{w}_k^{\ell}\mathrm{w}_{k-1})) \asymp \mathrm{cont}(\mathrm{w}_k^{\ell+1}\mathrm{w}_{k-1}),$$

where the last part comes from Proposition 3.11. On the other hand, $\mathrm{cont}(\mathrm{w}_k)\leq\mathrm{cont}(\mathrm{U}(\mathrm{w}_k))$ and $\mathrm{cont}(\mathrm{w}_k^{\ell}\mathrm{w}_{k-1})\leq\mathrm{cont}(\mathrm{U}(\mathrm{w}_k^{\ell}\mathrm{w}_{k-1}))$. So the left-hand side of (28) implies its right-hand side, and it just remains to prove the estimates with Y_i. For each $j\geq 0$, we set $c_j:=\mathrm{cont}(\mathrm{w}_j)$. By (27) we have $c_k=W_k^{\varrho/\beta+o(1)}$. Since $\mathrm{cont}(\mathrm{ww}')\geq\mathrm{cont}(\mathrm{w})\mathrm{cont}(\mathrm{w}')$ for each $\mathrm{w},\mathrm{w}'\in\mathrm{Mat}_{2\times 2}(\mathbb{Z})\setminus\{0\}$, we find

$$\mathrm{cont}(\mathrm{w}_k^{\ell+1}\mathrm{w}_{k-1}) \geq c_k^{\ell+1}c_{k-1} = (W_k^{\ell+1}W_{k-1})^{\varrho/\beta+o(1)}. \tag{34}$$

First case. Suppose that $\mathbf{s}$ is bounded. Then $o(s_{k+1}) = o(1)$ and $W_{k+1}^{o(1)} = W_k^{o(1)}$. We deduce easily (28) from (34) and (27) combined with the upper bound

$$c_{k+1} \geq c_k^{s_{k+1}-\ell-1}\mathrm{cont}(\mathrm{w}_k^{\ell+1}\mathrm{w}_{k-1}).$$

Second case. Suppose that $\mathbf{s}$ is unbounded and $\alpha = 2\varrho$. The matrix $\mathrm{w} := c^{-1}\mathrm{w}_k^{\ell+1}\mathrm{w}_{k-1}$ has integer coefficients, where $c := \mathrm{cont}(\mathrm{w}_k^{\ell+1}\mathrm{w}_{k-1})$. Using (26) we find

$$1 \leq |\det(\mathrm{w})| \leq c^{-2}(W_k^{\ell+1}W_{k-1})^{\alpha/\beta+o(1)} = c^{-2}(W_k^{\ell+1}W_{k-1})^{2\varrho/\beta+o(1)},$$

hence the upper bound $\mathrm{cont}(\mathrm{w}_k^{\ell+1}\mathrm{w}_{k-1}) \leq Y_i^{\varrho/\beta+o(1)}$. We conclude by (34). □

4.3. **Property of the set *Sturm*.**

Let $\mathbf{s}$ be a sequence of positive integers and set $\sigma = \sigma(\mathbf{s})$. It is difficult to compute the Diophantine exponents of an element of *Sturm*($\mathbf{s}$) using only Definition 1.9. The main result of this section, namely Theorem 4.11 below, will help dealing with this problem. Recall that $(p_k)_{k\geq -1}$ is defined by (25).

Theorem 4.11. *Let $\xi \in \mathrm{Sturm}(\mathbf{s})$ and set $\Xi = (1, \xi, \xi^2)$. Then, there exist $\beta, \delta \in \mathbb{R}$ with $\beta > 0$ and $\delta \in [0, \sigma/(1+\sigma)]$ with the following properties. We define $(W_k)_{k\geq 0} := (e^{\beta p_k})_{k\geq 0}$ and for each $i = t_k + \ell$ with $k \geq 1$ and $0 \leq \ell < s_{k+1}$, we set*

$$Y_i := W_k^{\ell+1}W_{k-1} \quad \text{and} \quad Z_i = W_k^{\ell}W_{k-1}.$$

Then, there is an admissible ψ-Sturmian sequence $(\mathrm{w}_k)_{k\geq 0}$ in $\mathrm{GL}_2(\mathbb{R})$, with multiplicative growth and defined over $\mathbb{Q}$, such that

(i) $\|\mathrm{w}_k\| \asymp W_k$ *and* $|\det(\mathrm{w}_k)| \asymp W_k^{\delta}$*;*

(ii) $\mathrm{cont}(\mathrm{w}_k) = W_k^{o(1)}$ *as k tends to infinity.*

Moreover, denoting for each $i \geq -1$ by $\mathbf{y}_i, \mathbf{z}_i \in \mathbb{Z}^3$ the non-zero primitive integer points

$$\mathbf{y}_i := \mathrm{cont}(\mathbf{a}_i)^{-1}\mathbf{a}_i \quad \text{and} \quad \mathbf{z}_i := \mathrm{cont}(\mathbf{b}_i)^{-1}\mathbf{b}_i, \tag{35}$$

where $(\mathbf{a}_i)_{i\geq -1}$ and $(\mathbf{b}_i)_{i\geq -1}$ are defined as in Definition 3.10, we have the estimates

$$\|\mathbf{y}_i\| = Y_i^{1+o(1)}, \quad \|\mathbf{y}_i \wedge \Xi\| = Y_i^{-1+\delta+o(1)}, \quad \|\mathbf{z}_i\| = Z_i^{1+o(1)} \quad \text{and} \quad |\mathbf{z}_i \cdot \Xi| = Y_i^{\delta+o(1)}Y_{i+1}^{-1} \tag{36}$$

as i tends to infinity. In particular, the sequence $(\mathbf{y}_i)_{i\geq -2}$ converges projectively to Ξ.

Remark. As we will see, the parameter δ in our theorem depends only on ξ (see the remark after Theorem 5.9).

Proof of Theorem 4.11. By Definition 1.9 (and the remarks below Definition 1.9), there is a sequence $(\mathbf{v}_i)_{i\geq i_0}$ (with $i_0 \in \mathbb{N}$) of non-zero primitive points in $\mathbb{Z}^3$ such that $\det(\mathbf{v}_i) \neq 0$ for each $i \geq i_0$, the point $\mathbf{v}_{i+1}$ is proportional to $\mathbf{v}_i\mathbf{v}_{\psi(i)}^{-1}\mathbf{v}_i$ for each i with $\psi(i) \geq i_0$, and

$$|\det(\mathbf{v}_i)| = \|\mathbf{v}_i\|^{\sigma/(1+\sigma)+o(1)} \tag{37}$$

as i tends to infinity with $\psi(i+1) < i$. The space generated by $(\mathbf{v}_i)_{i\geq i_0}$ has dimension 3. Indeed, if $(\mathbf{v}_i)_{i\geq i_0}$ was included in a subspace V of dimension 2 and defined over $\mathbb{Q}$, it would imply that $\Xi \in V$, which is impossible since the coordinates of Ξ are linearly independent over $\mathbb{Q}$. Upon defining $\mathbf{v}_{i_0-1}, \dots, \mathbf{v}_{-2}$ by using the induction formula $\mathbf{v}_{\psi(i)} = \mathbf{v}_i\mathbf{v}_{i+1}^{-1}\mathbf{v}_i$, we may assume without loss of generality that $i_0 = -2$. Then, by Proposition 3.6, there is an admissible ψ-Sturmian sequence $(\mathrm{w}'_k)_{k\geq 0}$ defined over $\mathbb{Q}$ and such that $\mathbf{y}'_i$ is proportional to $\mathbf{v}_i$ for each $i \geq -2$, where $(\mathbf{y}'_i)_{i\geq -2}$ denotes the sequence associated to $(\mathrm{w}'_k)_{k\geq 0}$ by Definition 3.4. According to Proposition 4.2 the sequence $(\mathrm{w}'_k)_{k\geq 0}$ has multiplicative growth. Any ψ-Sturmian sequence of first terms $\lambda\mathrm{w}'_0$ and $\mu\mathrm{w}'_1$ (with $\lambda, \mu \neq 0$) has the above properties. So, upon replacing w_0, w_1 by larger multiples, we may assume that $|\det(\mathrm{w}_0)|, |\det(\mathrm{w}_1)| > 1$, which implies that $(|\det(\mathrm{w}'_k)|)_{k\geq 0}$ tends to infinity, and therefore $(\mathrm{w}'_k)_{k\geq 0}$ is unbounded. Then, by Proposition 4.6, there are $\alpha, \beta', \varrho \in \mathbb{R}$ with $\beta' > 0$, such that

$$|\det(\mathrm{w}'_k)| \asymp e^{\alpha p_k}, \quad \|\mathrm{w}'_k\| \asymp e^{\beta' p_k}, \quad \mathrm{cont}(\mathrm{w}'_k) = e^{p_k(\varrho+o(1))}$$

as k tends to infinity. since $(\mathbf{y}'_i)_{i\geq -2}$ converges projectively to Ξ, if follows from the classical estimates of the determinant (see Section 3 of [15]) that

$$|\det(\mathbf{y}'_i)| \ll \|\mathbf{y}'_i\|\|\mathbf{y}'_i \wedge \Xi\| = o(\|\mathbf{y}'_i\|^2).$$

By taking $i = \psi(t_{k+1})$ and by using (16), we obtain $\det(\mathrm{w}'_k) = o(\|\mathrm{w}'_k\|^2)$, and thus $\alpha < 2\beta'$. Moreover, by definition of the content, the matrix $\mathrm{cont}(\mathrm{w}'_k)^{-1}\mathrm{w}'_k$ has integer coefficients, so its determinant is a non-zero integer. Consequently, we have $2\varrho \leq \alpha$. This leads us to $\beta := \beta' - \varrho > 0$. For each $k \geq 0$, we set

$$\mathrm{w}_k := e^{-\varrho p_k}\mathrm{w}'_k.$$

Then $(\mathrm{w}_k)_{k\geq 0}$ is a ψ-Sturmian sequence in $\mathrm{GL}_2(\mathbb{R})$ defined over $\mathbb{Q}$, admissible, with multiplicative growth, and we have

$$|\det(\mathrm{w}_k)| \asymp e^{(\alpha-2\varrho)p_k}, \quad \|\mathrm{w}_k\| \asymp e^{\beta p_k}, \quad \mathrm{cont}(\mathrm{w}_k) = e^{o(p_k)} = \|\mathrm{w}_k\|^{o(1)},$$

in particular $(\|\mathrm{w}_k\|)_{k\geq 0}$ tends to infinity since $\beta > 0$, and $|\det(\mathrm{w}_k)| \asymp \|\mathrm{w}_k\|^\delta$ with $\delta := (\alpha - 2\varrho)/\beta \in [0, 2)$. It proves (i) and (ii). Note that by proposition 3.11, up to multiplication by a constant, the sequences $(\mathbf{a}_i)_{i\geq -1}$ and $(\mathbf{b}_i)_{i\geq -1}$ are the sequences of Definition 3.4. Consequently

$$|\det(\mathbf{a}_i)| \asymp |\det(\mathrm{w}_k)|^{\ell+1}|\det(\mathrm{w}_{k-1})| \tag{38}$$

for $i = t_k + \ell$ with $k \geq 1$ and $0 \leq \ell < s_{k+1}$, and the point $\mathbf{a}_i$ is proportional to $\mathbf{y}'_i$ (and thus to $\mathbf{v}_i$). We deduce that $(\mathbf{a}_i)_{i\geq 0}$ converges projectively to Ξ, and Proposition 4.5 yields

$$\|\mathbf{a}_i \wedge \Xi\| \asymp \frac{|\det(\mathbf{a}_i)|}{\|\mathbf{a}_i\|} \quad \text{and} \quad |\mathbf{b}_i \cdot \Xi| \asymp \frac{|\det(\mathbf{a}_i)|}{\|\mathbf{a}_{i+1}\|}.$$

By (i) and (38), we have $|\det(\mathbf{a}_i)| = Y_i^{\delta+o(1)}$. Also note that $Y_{i+1}^{o(1)} = Y_i^{o(1)}$ since $Y_{i+1} = W_k Y_i \leq Y_i^2$. Putting the above estimates together with

$$\|\mathbf{a}_i\| = Y_i^{1+o(1)}, \quad \|\mathbf{b}_i\| = Z_i^{1+o(1)}, \quad \mathrm{cont}(\mathbf{a}_i) = Y_i^{o(1)} \quad \text{and} \quad \mathrm{cont}(\mathbf{b}_i) = Z_i^{o(1)}$$

coming from Proposition 4.6, we get (36). Note that $\mathbf{y}_i = \pm\mathbf{v}_i$ since they are linearly dependent primitive integer points. We obtain $\delta \leq \sigma/(1+\sigma)$ by combining (37) with

$$|\det(\mathbf{v}_i)| = |\det(\mathbf{y}_i)| \ll \|\mathbf{y}_i \wedge \Xi\|\|\mathbf{y}_i\| = Y_i^{\delta+o(1)} = \|\mathbf{v}_i\|^{\delta+o(1)}.$$

$\square$

Now, let us briefly recall the definition of Sturmian type numbers constructed in [24].

Definition 4.12. Suppose $\mathbf{s}$ bounded. A *proper ψ-Sturmian number* is a real number ξ such that there are a real number $\delta \in [0, \sigma/(1+\sigma))$ and an admissible ψ-Sturmian sequence $(\mathrm{w}_k)_{k\geq 0}$ of matrices in $\mathrm{GL}_2(\mathbb{Q}) \cap \mathrm{Mat}_{2\times 2}(\mathbb{Z})$ with the following properties. The sequence of symmetric matrices $(\mathbf{y}_i)_{i\geq -2}$ associated to $(\mathrm{w}_k)_{k\geq 0}$ by Definition 3.4 converges projectively to $(1, \xi, \xi^2)$ and $(\mathrm{cont}(\mathbf{y}_i))_{i\geq -2}$ is bounded. Moreover $(\mathrm{w}_k)_{k\geq 0}$ is unbounded, has multiplicative growth and satisfies $|\det(\mathrm{w}_k)| \asymp \|\mathrm{w}_k\|^\delta$. The set of *Sturmian type numbers* is the union of the sets of proper $\psi_{\mathbf{s}}$-Sturmian numbers for bounded sequences $\mathbf{s}$.

Remark. The elements of *Sturm*$(\mathbf{s})$ (when $\mathbf{s}$ is bounded and $\delta < \sigma/(1+\sigma)$) have a lot in common with proper ψ-Sturmian numbers. However, a major difference is the possible existence of non-trivial contents for the sequences involved in Theorem 4.11. Also note that in our theorem, $(\mathrm{w}_k)_{k\geq 0}$ does not necessarily have integer coefficients.

5. Applications to Diophantine approximation

Our proof of Theorem 1.3 (see §5.4) relies on parametric geometry of numbers. We recall the elements of the theory that we need in §5.1, and in §5.2 we compute the parametric versions of the exponents $\widehat{\omega}$, ω, $\widehat{\lambda}$, λ, associated to a point $\xi \in \mathit{Sturm}(\mathbf{s})$ and $\widehat{\lambda}_{\min}(\xi)$. The two remaining exponents $\widehat{\omega}_2^*$ and ω_2^* are studied separately in §5.3.

5.1. **Parametric geometry of numbers.**

Let $\Xi = (1, \xi, \xi^2)$ where $\xi \in \mathbb{R}$ is neither rational nor quadratic. In this section we quickly present Schmidt and Summerer's tools of parametric geometry of numbers in dimension 3 (see [36] and [37]). In the following, the letter q always denotes a positive real number. Our setting is the same as that of [24], *i.e.* we consider the two following families of symmetric convex bodies:

$$\mathcal{C}_\xi(e^q) := \{\mathbf{x} \in \mathbb{R}^3 \ ; \ \|\mathbf{x}\| \leq 1 \text{ and } |\mathbf{x} \cdot \Xi| \leq e^{-q}\}$$

and

$$\mathcal{C}^*_\xi(e^q) := \{\mathbf{x} \in \mathbb{R}^3 \ ; \ \|\mathbf{x}\| \leq e^q \text{ and } \|\mathbf{x} \wedge \Xi\| \leq 1\}.$$

For $j = 1, 2, 3$, the quantity $\lambda_j(q)$ (resp. $\lambda_j^*(q)$) denotes the j-th successive minimum of the convex body $\mathcal{C}_\Xi(e^q)$ (resp. $\mathcal{C}^*_\Xi(e^q)$) with respect to the lattice $\mathbb{Z}^3$. We also define

$$\mathrm{L}_j(q) = \log \lambda_j(q), \quad \psi_j(q) = \frac{\mathrm{L}_j(q)}{q}, \quad \overline{\psi}_j = \limsup_{q\to\infty} \psi_j(q), \quad \underline{\psi}_j = \liminf_{q\to\infty} \psi_j(q),$$

as well as the analogous quantities $\mathrm{L}_j^*(q)$, $\psi_j^*(q)$, $\overline{\psi}_j^*$, $\underline{\psi}_j^*$ associated to $\lambda_j^*(q)$. We group these successive minima L_j (resp. L_j^*) into a single map $\mathbf{L}_\xi = (\mathrm{L}_1, \mathrm{L}_2, \mathrm{L}_3)$ (resp. $\mathbf{L}_\xi^* = (\mathrm{L}_1^*, \mathrm{L}_2^*, \mathrm{L}_3^*)$). In the following proposition (cf [36] and [31]) we give a classical relation between standard and parametric Diophantine exponents.

Proposition 5.1. *Let ξ be a real number which is neither rational nor quadratic. Then*

$$(\underline{\psi}_1, \overline{\psi}_1, \underline{\psi}_3, \overline{\psi}_3) = \Big(\frac{1}{\omega_2(\xi)+1}, \frac{1}{\widehat{\omega}_2(\xi)+1}, \frac{\widehat{\lambda}_2(\xi)}{\widehat{\lambda}_2(\xi)+1}, \frac{\lambda_2(\xi)}{\lambda_2(\xi)+1}\Big). \tag{39}$$

Note that there also exists a parametric version of $\widehat{\lambda}_{\min}(\xi)$ (see [25, Section 3.2 and Proposition 3.6]), but we will not need it here.

We follow [37, §3] and we define the *combined graph* of a set of real valued functions defined on an interval I to be the union of their graphs in $I \times \mathbb{R}$. For a map $\mathbf{P} : [c, +\infty) \to \mathbb{R}^3$ and an interval $I \subseteq [c, +\infty)$, we also defined the *combined graph of* $\mathbf{P}$ *on* I to be the combined graph of its components P_1, P_2, P_3 restricted to I. In order to study the combined graph of the map $\mathbf{L}_\xi$, it is useful to define the following functions.

Definition 5.2. For each point $\mathbf{x} \in \mathbb{R}^n \setminus \{0\}$ we denote by $\lambda_\mathbf{x}(q)$ (resp. $\lambda_\mathbf{x}^*(q)$) the smallest real number $\lambda > 0$ such that $\mathbf{x} \in \lambda\mathcal{C}_\Xi(e^q)$ (resp. $\mathbf{x} \in \lambda\mathcal{C}^*_\Xi(e^q)$). Then, we set

$$\mathrm{L}_\mathbf{x}(q) = \log(\lambda_\mathbf{x}(q)) \quad \text{and} \quad \mathrm{L}_\mathbf{x}^*(q) = \log(\lambda_\mathbf{x}^*(q)).$$

Roy calls the graph of $\mathrm{L}_\mathbf{x}$ (or of $\mathrm{L}_\mathbf{x}^*$) the *trajectory* of $\mathbf{x}$.

Locally, the combined graph of $\mathbf{L}_\xi$ is included in the combined graph of a finite set $\mathrm{L}_\mathbf{x}$, and for each $\mathbf{x} \neq 0$ we have

$$\mathrm{L}_\mathbf{x}(q) = \max\big(\log(\|\mathbf{x}\|), \log(|\mathbf{x}\cdot\Xi|) + q\big) \quad \text{and} \quad \mathrm{L}_\mathbf{x}^*(q) = \max\big(\log(\|\mathbf{x}\wedge\Xi\|), \log(\|\mathbf{x}\|) - q\big).$$

Note that

$$\mathrm{L}_1(q) = \min_{\mathbf{x}\neq 0} \mathrm{L}_\mathbf{x}(q) \quad \text{and} \quad \mathrm{L}_1^*(q) = \min_{\mathbf{x}\neq 0} \mathrm{L}_\mathbf{x}^*(q).$$

Proposition 5.3 (Mahler)**.** *For each $j = 1, 2, 3$, we have $\underline{\psi}_j = -\overline{\psi}_{4-j}$ and $\overline{\psi}_j = -\underline{\psi}_{4-j}$. More precisely* $\mathrm{L}_j(q) = -\mathrm{L}_{4-j}^*(q) + \mathcal{O}(1)$ *for each $q > 0$.*

The functions L_j have many rigid properties. For example they are continuous, piecewise linear with slopes 0 and 1, and by Minkowski's second Theorem, for any $q \geq 0$ we have

$$\mathrm{L}_1(q) + \mathrm{L}_2(q) + \mathrm{L}_3(q) = q + \mathcal{O}(1). \tag{40}$$

To describe precisely their behavior, several class of functions have been introduced, starting with the model of (n,γ)-systems of Schmidt and Summerer in [37]. In [24], we use the simpler notion of n-system given by Roy in [32]. The main result of [31] implies that $\mathbf{L}_\xi$ can be approximated, up to an additive constant, by a 3-system, and vice versa.

5.2. **Map of the successive minima.**

Let $\xi \in \mathit{Sturm}(\mathbf{s})$, where $\mathbf{s}$ is a sequence of positive integers. The goal of this section is to describe the map of successive minima $\mathbf{L}_\xi$ and to determine its parametric exponents, see Proposition 5.8 and Theorem 5.9 respectively. Our strategy is to construct a simpler and explicit function $\mathbf{P}$ (very similar to that in [24, Section 7.2]) and show that $\mathbf{P}(q) = \mathbf{L}_\xi(q) + o(q)$, except in some controlled intervals which may be ignored for the computation of the parametric exponents, as was already the case in [24]. Note that in [24], the sequence $\mathbf{s}$ is bounded and the parameter δ is $< \sigma/(1+\sigma)$. Here, $\mathbf{s}$ might be unbounded and we allow the case $\delta = \sigma/(1+\sigma)$, which brings some technical difficulties.

Set $\psi = \psi_{\mathbf{s}}$ and $\sigma = \sigma(\mathbf{s})$ and let $\delta \in [0, \sigma/(1+\sigma)]$, $(W_k)_{k\geq 0}$, $(Y_i)_{i\geq 0}$, $(Z_i)_{i\geq 0}$, and the sequences of primitive integer points $(\mathbf{y}_i)_{i\geq -1}$ and $(\mathbf{z}_i)_{i\geq 0}$ in $\mathbb{Z}^3$ be as in Theorem 4.11. Note that $W_k > 1$ for each $k \geq 0$ and that $(Y_i)_{i\geq 0}$ is increasing. The theory of continued fractions yields

$$\frac{\log W_k}{\log W_{k+1}} = \frac{p_k}{p_{k+1}} = \frac{1}{[s_{k+1}; s_k, \dots, s_1]} \leq \frac{1}{\gamma} + o(1) \tag{41}$$

as $k \geq 0$ tends to infinity. We also get the useful formula (see [24, Eq. (7.2)])

$$\sigma = \liminf_{k\to\infty} \frac{p_k}{p_{k+1}} = \liminf_{k\to\infty} \frac{\log W_k}{\log W_{k+1}}. \tag{42}$$

Definition 5.4. Given $i = t_k + \ell$ with $k \geq 1$ and $0 \leq \ell < s_{k+1}$, we denote by δ_i the maximum of the real numbers $\eta \leq \delta$ such that $Y_i^{1-\eta} \geq \max(Z_{t_{k+1}}, Z_i)$. We set $q_i = (2-\delta_i)\log(Y_i)$ and $c_i = q_i + \log(W_k)$, as well as

$$\Delta_i^* := \frac{Y_i^{\delta_i}}{Y_i} = \left(W_k^{\ell+1} W_{k-1}\right)^{-(1-\delta_i)} \quad \text{and} \quad \Delta_i := \frac{Y_i^{\delta_i}}{Y_{i+1}} = \frac{\Delta_i^*}{W_k}.$$

The functions $\widehat{\mathrm{L}}_i$ and $\widehat{\mathrm{L}}_i^*$ are defined for each $q \geq 0$ by

$$\widehat{\mathrm{L}}_i(q) = \max\left(\log(Z_i), \log(\Delta_i) + q\right) \quad \text{and} \quad \widehat{\mathrm{L}}_i^*(q) = \max\left(\log(\Delta_i^*), \log(Y_i) - q\right).$$

Since $Y_i > \max(Z_{t_{k+1}}, Z_i)$, we have $\delta_i \geq 0$. Note that $q_i = \log Y_i - \log \Delta_i^* = \log Z_i - \log \Delta_i$ is the point at which $\widehat{\mathrm{L}}_i$ and $\widehat{\mathrm{L}}_i^*$ change slope.

Lemma 5.5. *The sequence $(\delta_i)_{i\geq 0}$ converges to δ and $q_i < c_i < q_{i+1}$ for each large enough i.*

Remark. If $\delta < \sigma/(1+\sigma)$, then we have $\delta_i = \delta$ for each i large enough by [24, Proposition 7.17]. However, this might not be true if $\delta = \sigma/(1+\sigma) > 0$.

Proof. Let $i = t_k + \ell$ with $k \geq 1$ and $0 \leq \ell < s_{k+1}$. Recall that $Y_{i+1} = W_k Y_i$. If $\delta = 0$, then $\delta_i = 0$ for each $i \geq 0$ and we obtain $q_{i+1} = q_i + 2\log W_k > c_i > q_i$. Suppose now that $\delta > 0$, and therefore $\sigma > 0$. Using the inequality $\log W_{k-1} \geq (\sigma - o(1))\log W_k$ coming from (42), we find

$$\log Y_i - (1+\sigma)\log Z_{t_{k+1}} = (\ell+1)\log W_k + \log W_{k-1} - (1+\sigma)\log W_k \geq o(\log W_k).$$

We deduce from the above and $1-\delta \geq 1/(1+\sigma)$ that $Y_i^{1-\delta+o(1)} \geq Z_{t_{k+1}}$. Similarly,

$$\begin{aligned} Y_i-(1+\sigma)\log Z_i &= (\ell+1)\log W_k+\log W_{k-1}-(1+\sigma)(\ell\log W_k+\log W_{k-1}) \\ &\geq (s_{k+1}+1)\log W_k+\log W_{k-1}-(1+\sigma)(s_{k+1}\log W_k+\log W_{k-1}) \\ &= \log W_k-\sigma\log W_{k+1} \geq o(\log W_{k+1}), \end{aligned}$$

from which we get $Y_i^{1-\delta} \geq Z_i W_{k+1}^{o(1)}$. The sequence $\mathbf{s}$ is bounded since $\sigma>0$, so that $o(s_{k+1})=o(1)$ and $W_{k+1}^{o(1)}=Y_i^{o(1)}=W_k^{o(1)}$. We thus have $Y_i^{1-\delta+o(1)} \geq \max(Z_{t_{k+1}}, Z_i)$, hence $\delta_i=\delta+o(1)$. As a consequence if k is large enough, then $q_{i+1}=q_i+(2-\delta+o(1))\log W_k > c_i > q_i$. □

Lemma 5.6. *There exists an index $i_0 \geq 0$ such that for each $i=t_k+\ell \geq i_0$ with $k \geq 1$ and $0 \leq \ell < s_{k+1}$, the combined graph of $\widehat{\mathrm{L}}_{t_{k+1}}$, $-\widehat{\mathrm{L}}_i^*$, $\widehat{\mathrm{L}}_i$ on $[c_{i-1},c_i]$ is as on Figure 1. Furthermore*

$$\widehat{\mathrm{L}}_i^*(c_i)=\widehat{\mathrm{L}}_{i+1}^*(c_i) \quad \textit{and} \quad \widehat{\mathrm{L}}_i(c_i)=\begin{cases} \widehat{\mathrm{L}}_{i+1}(c_i) & \textit{if } \ell<s_{k+1}-1 \\ \widehat{\mathrm{L}}_{t_{k+2}}(c_i) & \textit{if } \ell=s_{k+1}-1. \end{cases} \tag{43}$$

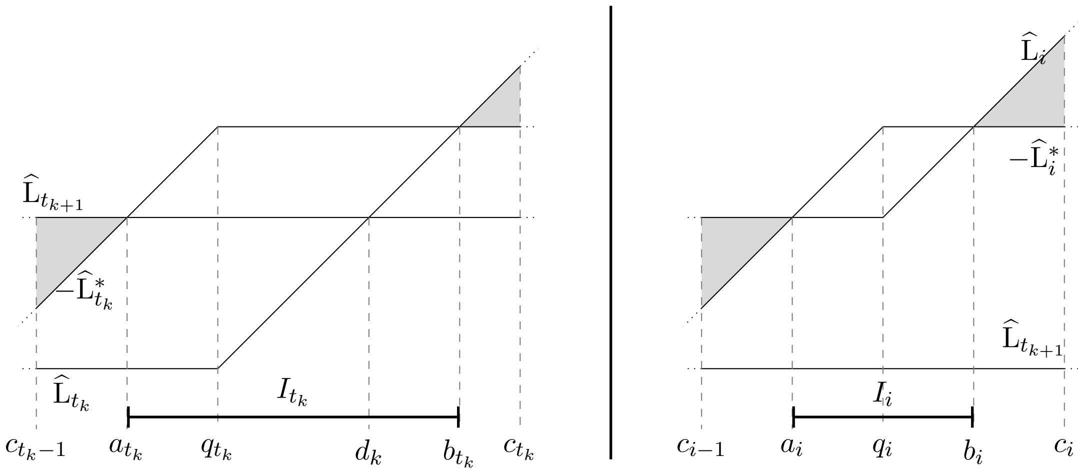

FIGURE 1. Combined graph of $\widehat{\mathrm{L}}_{t_{k+1}}$, $-\widehat{\mathrm{L}}_i^*$, $\widehat{\mathrm{L}}_i$ on $[c_{i-1},c_i]$ with $t_k<i<t_{k+1}$

Proof. By Lemma 5.5 we can suppose i large enough so that $c_{j-1}<q_j<c_j$ for each $j \geq i$. Recall that q_i is the point at which $\widehat{\mathrm{L}}_i$ and $\widehat{\mathrm{L}}_i^*$ change slope and that $Y_{i+1}=W_kY_i$. The intersection point abscissa of $\widehat{\mathrm{L}}_i^*$ and $\widehat{\mathrm{L}}_{i+1}^*$ is $\log Y_{i+1}-\log\Delta_i^*=c_i$. If $\ell<s_{k+1}-1$ (resp. $\ell=s_{k+1}-1$) then $Z_i<Z_{i+1}=Y_i$ (resp. $Z_i<Z_{t_{k+2}}=Y_i$) and the intersection point abscissa of $\widehat{\mathrm{L}}_i$ and $\widehat{\mathrm{L}}_{i+1}$ (resp. $\widehat{\mathrm{L}}_i$ and $\widehat{\mathrm{L}}_{t_{k+2}}$) is $\log Y_i-\log\Delta_i=c_i$. Hence (43).

We now prove the first part of the lemma. It suffices to compare $\widehat{\mathrm{L}}_{t_{k+1}}(q)$, $-\widehat{\mathrm{L}}_i^*(q)$ and $\widehat{\mathrm{L}}_i(q)$ at $q=c_{i-1}$, q_i and c_i. By definition of δ_i, we have

$$(1-\delta_i)\log Y_i=-\widehat{\mathrm{L}}_i^*(q_i) \geq \max(\widehat{\mathrm{L}}_i(q_i),\widehat{\mathrm{L}}_{t_{k+1}}(q_i))=\begin{cases} \widehat{\mathrm{L}}_i(q_i)=\log Z_i & \text{if } i>t_k \\ \widehat{\mathrm{L}}_{t_{k+1}}(q_i)=\log W_k & \text{if } i=t_k. \end{cases}$$

Since $-\widehat{\mathrm{L}}_i^*$, $\widehat{\mathrm{L}}_{t_{k+1}}$ are constant on $[q_i,c_i]$, we deduce that $-\widehat{\mathrm{L}}_i^*(c_i) \geq \widehat{\mathrm{L}}_{t_{k+1}}(c_i)$. Moreover

$$\widehat{\mathrm{L}}_i(c_i)=\widehat{\mathrm{L}}_i(q_i)+c_i-q_i=\log Y_i \geq (1-\delta_i)\log Y_i=-\widehat{\mathrm{L}}_i^*(c_i) \geq \widehat{\mathrm{L}}_{t_{k+1}}(c_i).$$

If $i>t_k$, then by the above, we have $\widehat{\mathrm{L}}_{i-1}(c_{i-1}) \geq -\widehat{\mathrm{L}}_{i-1}^*(c_{i-1}) \geq \widehat{\mathrm{L}}_{t_{k+1}}(c_{i-1})$. Combined with (43), this is equivalent to

$$\widehat{\mathrm{L}}_i(c_{i-1}) \geq -\widehat{\mathrm{L}}_i^*(c_{i-1}) \geq \widehat{\mathrm{L}}_{t_{k+1}}(c_{i-1}).$$

Similarly, for $i = t_{k-1} + s_k - 1 = t_k - 1$, Eq. (43) together with $\widehat{\mathrm{L}}_i(c_i) \geq -\widehat{\mathrm{L}}_i^*(c_i) \geq \widehat{\mathrm{L}}_{t_k}(c_i)$ yields

$$\widehat{\mathrm{L}}_{t_{k+1}}(c_{t_k-1}) \geq -\widehat{\mathrm{L}}_{t_k}^*(c_{t_k-1}) \geq \widehat{\mathrm{L}}_{t_k}(c_{t_k-1}).$$

□

Definition 5.7. Let $i_0 \geq 0$ satisfying Lemma 5.6. We define the function $\mathbf{P} = (\mathrm{P}_1, \mathrm{P}_2, \mathrm{P}_3)$ on $[c_{i_0-1}, +\infty)$ as follows. For integers $i, k \geq 0$ with $i \geq i_0$ and $t_k \leq i < t_{k+1}$, we set for each $q \in (c_{i-1}, c_i]$

$$\mathbf{P}(q) := \Phi\Big(\widehat{\mathrm{L}}_{t_{k+1}}(q), -\widehat{\mathrm{L}}_i^*(q), \widehat{\mathrm{L}}_i(q)\Big),$$

where $\Phi : \mathbb{R}^3 \to \mathbb{R}^3$ is the function which lists the coordinates of a point in monotonically increasing order. For each $i \geq i_0$ we denote by $I_i = [a_i, b_i] \ni q_i$ the subinterval of $[c_{i-1}, c_i]$ on which $\mathrm{P}_3 = -\widehat{\mathrm{L}}_i^*$, and we set $I'_i = [b_i, a_{i+1}] \ni c_i$, see Figure 1.

If $\delta_i = 0$, then $I'_i = \{c_i\}$. By Lemma 5.6 the function $\mathbf{P}$ is continuous, $\mathrm{P}_1 \leq \mathrm{P}_2 \leq \mathrm{P}_3$ and

$$\mathrm{P}_1(q) + \mathrm{P}_2(q) + \mathrm{P}_3(q) = q \tag{44}$$

for each $q \geq c_{i_0}$. More generally, we can show that $\mathbf{P}$ is a 3-system on $[c_{i_0}, +\infty)$ (as defined in [24, Definition 7.9]), whose combined graph is as that in Figure 2.

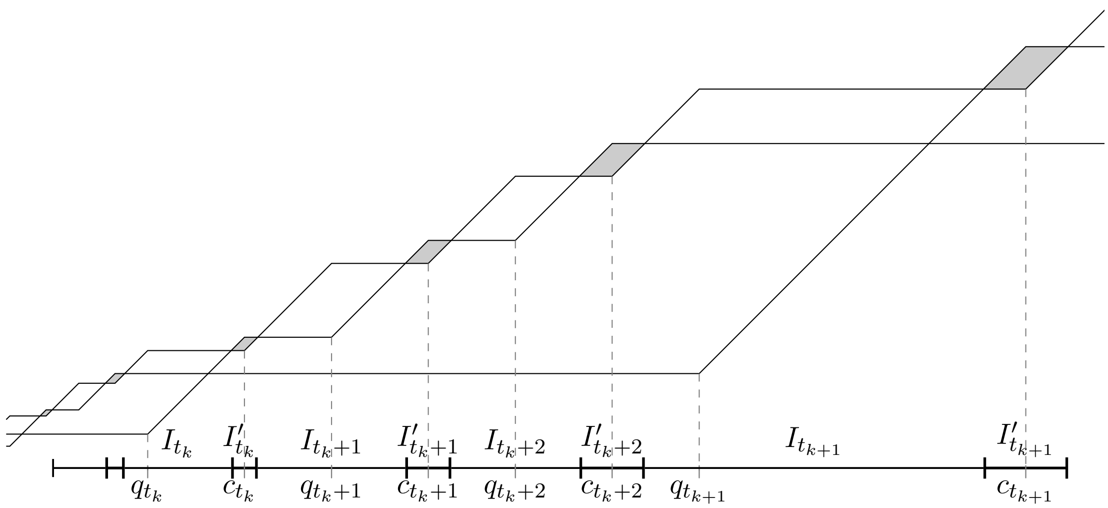

FIGURE 2. Combined graph of $\mathbf{P}$ (with $s_{k+1} = 3$)

Using the estimate $\delta_i = \delta + o(1)$ together with (36), we get the following estimates

$$\|\mathbf{y}_i\| = Y_i^{1+o(1)}, \quad \|\mathbf{y}_i \wedge \Xi\| = \Delta_i^* Y_i^{o(1)}, \quad \|\mathbf{z}_i\| = Z_i^{1+o(1)} \quad \text{and} \quad |\mathbf{z}_i \cdot \Xi| = \Delta_i Y_i^{o(1)}, \tag{45}$$

as i tends to infinity. They play a crucial role in the proof of our next result.

Proposition 5.8. *Let $\xi \in \mathrm{Sturm}(\mathbf{s})$ and denote by $\mathbf{P}$ the function associated to ξ as in Definition 5.7. Set $I := \bigcup_{i \geq i_0} I_i$ and $I' := \bigcup_{i \geq i_0} I'_i$. Then*

(i) *As $q \in I$ tends to infinity, we have $\mathbf{L}_\xi(q) = \mathbf{P}(q) + o(q)$;*
(ii) *As $q \in I'$ tends to infinity, we have $\mathrm{L}_1(q) = \mathrm{P}_1(q) + o(q)$ and*

$$\mathrm{P}_2(q) \leq \mathrm{L}_2(q) + o(q) \leq \frac{\mathrm{P}_2(q) + \mathrm{P}_3(q)}{2} \leq \mathrm{L}_3(q) + o(q) \leq \mathrm{P}_3(q). \tag{46}$$

In particular, if $\delta = 0$, then $\mathbf{L}_\xi(q) = \mathbf{P}(q) + o(q)$ as q tends to infinity.

Roughly speaking, the combined graph of L_2 and L_3 on I'_j is included –within $o(c_j)$ – in the corresponding shaded area on Figure 2. Our strategy is very similar to that in [24, proof of Proposition 7.20]. Here, the situation is a bit more complicated because $\mathbf{s}$ can be unbounded and we deal with some $o(q)$ instead of $\mathcal{O}(1)$.

Proof. Since $Y_{i+1}^{o(1)} = Y_i^{o(1)}$ as i tends to infinity, we have $o(q_{i+1}) = o(q_i)$ and thus $o(a_i) = o(b_i) = o(a_{i+1})$. Note that if $\delta = 0$, then $I'_i = \{c_i\}$ and (i) implies (ii) and the last part of the proposition.

Suppose that (i) holds, and let us prove (ii). Recall that L_1, L_2, L_3 are (continuous) piecewise linear with slope 0 or 1. In particular, they are monotonically increasing. Let $i > 0$ and $q \in I'_i = [b_i, a_{i+1}]$. We deduce from (i) the estimates

$$\mathrm{P}_1(b_i) + o(b_i) = \mathrm{L}_1(b_i) \leq \mathrm{L}_1(q) \leq \mathrm{L}_1(a_{i+1}) = \mathrm{P}_1(a_{i+1}) + o(a_{i+1}).$$

Since P_1 is constant on the interval I'_i and $o(a_{i+1}) = o(b_i)$, we obtain $\mathrm{L}_1(q) = \mathrm{P}_1(q) + o(q)$. The function P_3 is increasing with slope 1 on $[b_i, c_i]$, and constant on $[c_i, a_{i+1}]$. Since L_3 has slope at most 1 and satisfies $\mathrm{L}_3(b_i) = \mathrm{P}_3(b_i) + o(b_i)$ and $\mathrm{L}_3(a_{i+1}) = \mathrm{P}_3(a_{i+1}) + o(a_{i+1})$ by (i), we get $\mathrm{L}_3(q) \leq \mathrm{P}_3(q) + o(q)$. Similarly, we find $\mathrm{L}_2(q) \geq \mathrm{P}_2(q) + o(q)$. Finally, by (40), the estimate $\mathrm{L}_1(q) = \mathrm{P}_1(q) + o(q)$ and (44), we find

$$\mathrm{L}_2(q) + \mathrm{L}_3(q) = q - \mathrm{L}_1(q) + \mathcal{O}(1) = \mathrm{P}_2(q) + \mathrm{P}_3(q) + o(q).$$

Combining the above with $\mathrm{L}_2(q) \leq \mathrm{L}_3(q)$, we obtain the remaining inequalities of (46).

We now prove (i), or equivalently, that for each $\varepsilon > 0$ there exists i_1 such that

$$\|\mathbf{L}_\xi(q) - \mathbf{P}(q)\| \leq \varepsilon q \tag{47}$$

for each $q \in I_i$ with $i \geq i_1$. Let $i > i_0$ and $k \geq 0$ such that $t_k \leq i < t_{k+1}$. Eq. (45) implies that

$$\mathbf{P}(q) = \Phi\big(\mathrm{L}_{\mathbf{z}_{t_{k+1}}}(q), \mathrm{L}_{\mathbf{z}_i}(q), -\mathrm{L}^*_{\mathbf{y}_i}(q)\big) + o(q) \qquad (q \in I_i), \tag{48}$$

where $\Phi : \mathbb{R}^3 \to \mathbb{R}^3$ is the function which lists the coordinates of a point in monotonically increasing order. The points $\mathbf{z}_i$ and $\mathbf{z}_{t_{k+1}}$ are linearly independent, for $\mathbf{z}_{t_{k+1}} \wedge \mathbf{z}_i = \pm \mathbf{y}_i$ by definition. This implies that $\mathrm{L}_1 \leq \min\{\mathrm{L}_{\mathbf{z}_i}, \mathrm{L}_{\mathbf{z}_{t_{k+1}}}\}$ and $\mathrm{L}_2 \leq \max\{\mathrm{L}_{\mathbf{z}_i}, \mathrm{L}_{\mathbf{z}_{t_{k+1}}}\}$. Similarly, we have $\mathrm{L}^*_1 \leq \mathrm{L}^*_{\mathbf{y}_i}$. Combined with (48) and $\mathrm{L}_3 = -\mathrm{L}^*_1 + \mathcal{O}(1)$, we obtain, as $q \in I_i$ tends to infinity,

$$\mathrm{L}_1(q) \leq \mathrm{P}_1(q) + o(q), \quad \mathrm{L}_2(q) \leq \mathrm{P}_2(q) + o(q) \quad \text{and} \quad \mathrm{L}_3(q) \geq \mathrm{P}_3(q) + o(q). \tag{49}$$

Fix $\varepsilon_1, \varepsilon_2 > 0$ and choose $q \in I_i$.

First case. Suppose that $\mathrm{L}_3(q) \leq \mathrm{P}_3(q) + \varepsilon_1 q$. By (40) and (44) we have

$$\mathrm{L}_1(q) + \mathrm{L}_2(q) + \mathcal{O}(1) = q - \mathrm{L}_3(q) \geq q - \mathrm{P}_3(q) - \varepsilon_1 q = \mathrm{P}_1(q) + \mathrm{P}_2(q) - \varepsilon_1 q.$$

Combined with (49), it shows that (47) holds with $\varepsilon = 2\varepsilon_1$ if i is large enough.

Second case. Suppose that $\mathrm{L}_3(q) \geq \mathrm{P}_3(q) + \varepsilon_1 q$. We claim that if q is large enough, then necessarily $i = t_k$ and $|q - d_k| < \varepsilon_2 d_k$ (where d_k is defined by $\widehat{\mathrm{L}}_{t_{k+1}}(d_k) = \widehat{\mathrm{L}}_{t_k}(d_k)$, see Figure 1). Yet the components of $\mathbf{L}_\xi$ and $\mathbf{P}$ are continuous with slope 0 or 1. So, by using the first case with $q = (1 - \varepsilon_2) d_k$ and by taking ε_2 sufficiently small, it yields (47) with $\varepsilon = 3\varepsilon_1$. We now prove our claim.

If q is large enough, and since $\mathrm{L}^*_1 = -\mathrm{L}_3 + \mathcal{O}(1)$ and $-\mathrm{P}_3(q) = \mathrm{L}^*_{\mathbf{y}_i}(q) + o(q)$, we deduce the existence of a non-zero primitive point $\mathbf{y} \in \mathbb{Z}^3$ such that $\mathrm{L}^*_1(q) = \mathrm{L}^*_{\mathbf{y}}(q) < \mathrm{L}^*_{\mathbf{y}_i}(q)$. It follows that $\mathbf{y}$ and $\mathbf{y}_i$ are linearly independent (since they are both primitive), hence $\mathrm{L}^*_2(q) \leq \mathrm{L}^*_{\mathbf{y}_i}(q)$. Combined with Mahler's duality and (48), we deduce that $\mathrm{L}_2(q) \geq \mathrm{P}_3(q) + o(q)$. In view of (49), we obtain $\mathrm{L}_2(q) = \mathrm{P}_2(q) + o(q) = \mathrm{P}_3(q) + o(q)$, from which we infer

$$\mathrm{L}_1(q) = q - \mathrm{L}_2(q) - \mathrm{L}_3(q) + \mathcal{O}(1) \leq q - \mathrm{P}_2(q) - \mathrm{P}_3(q) - \varepsilon_1 q + o(q) = \mathrm{P}_1(q) - \varepsilon_1 q + o(q).$$

Consequently, if q is large enough, then $\mathrm{L}_1(q) < \min\{\mathrm{L}_{\mathbf{z}_i}(q), \mathrm{L}_{\mathbf{z}_{t_{k+1}}}(q)\} = \mathrm{P}_1(q) + o(q)$. Since $\mathbf{z}_i$ and $\mathbf{z}_{t_{k+1}}$ are both primitive points, we obtain

$$\mathrm{L}_2(q) \le \min\{\mathrm{L}_{\mathbf{z}_i}(q), \mathrm{L}_{\mathbf{z}_{t_{k+1}}}(q)\} = \mathrm{P}_1(q) + o(q).$$

Hence $\mathrm{P}_1(q) = \mathrm{P}_2(q) + o(q) = \mathrm{P}_3(q) + o(q)$. Fix $\varepsilon_3 > 0$. Then, by the above and (44), there exists i_1 such that if $i \ge i_1$, then

$$|\mathrm{P}_j(q) - \frac{q}{3}| \le \varepsilon_3 q \quad (j = 1, 2, 3).$$

By taking ε_3 small enough, we deduce that $i = t_k$ and $q \in [(1-\varepsilon_2)d_k, (1+\varepsilon_2)d_k]$, for P_1 is increasing with slope 1 on $[q_{t_k}, d_k]$, and $\mathrm{P}_3 \ge (\mathrm{P}_2 + \mathrm{P}_3)/2$, which is increasing with slope $1/2$ on $[d_k, q_{t_{k+1}}]$ (see Figures 1 and 2). This ends the proof of our claim. □

As a consequence we get the following result.

Theorem 5.9. *Let $\xi \in \mathrm{S}turm(\mathbf{s})$ and denote by σ, δ the parameters associated to ξ as at the beginning of this section. For each $i = 1, 2, 3$, we denote by $\underline{\psi}_i, \overline{\psi}_i$ the parametric exponents associated to $\Xi = (1, \xi, \xi^2)$ as in Section 5.1. Then, we have*

$$\underline{\psi}_1 = \frac{\sigma}{(2-\delta)(1+\sigma)}, \quad \overline{\psi}_1 = \frac{1}{(1-\delta)(1+\sigma)+2}, \quad \overline{\psi}_2 = \frac{1}{2+\sigma},$$
$$\underline{\psi}_3 = \frac{(1-\delta)(1+\sigma)}{1+2(1-\delta)(1+\sigma)}, \quad \frac{1-\delta}{2-\delta} \le \overline{\psi}_3 \le \max\Big(\frac{1}{2-\delta+\sigma}, \frac{1-\delta}{2-\delta}\Big).$$

If δ satisfies the stronger condition $\delta < h(\sigma)$ with $h(\sigma) = \frac{\sigma}{2} + 1 - \sqrt{\left(\frac{\sigma}{2}\right)^2 + 1}$, then

$$\overline{\psi}_3 = \frac{1-\delta}{2-\delta} \quad \textit{and} \quad \widehat{\lambda}_{\min}(\xi) = \frac{(1-\delta)(1+\sigma)}{2+\sigma}.$$

The left-hand side equality still holds if $\delta = h(\sigma)$.

Remark. Theorem 5.9 shows that the parameters σ and

$$\delta(\xi) := \delta \tag{50}$$

depend only on ξ.

Proof of Theorem 5.9. Recall that $\mathbf{P} = (\mathrm{P}_1, \mathrm{P}_2, \mathrm{P}_3)$ is the function introduced in Definition 5.7. We define the parametric exponents associated to $\mathbf{P}$ by

$$\underline{\vartheta}_i := \liminf_{q\to\infty} \frac{\mathrm{P}_i(q)}{q} \quad \text{and} \quad \overline{\vartheta}_i := \limsup_{q\to\infty} \frac{\mathrm{P}_i(q)}{q} \quad (i = 1, 2, 3).$$

They are computed in [24] when $\mathbf{s}$ is bounded and $\delta < \sigma/(1+\sigma)$. The expressions for the exponents $\underline{\psi}_i, \overline{\psi}_i$ are obtained by using Proposition 5.8 and by arguing exactly as in the proof of [24, Theorem 7.2], so we will skip most of the details. Note that if $\mathbf{s}$ is unbounded, then $\delta = 0$ and $\mathbf{L}_\xi(q) = \mathbf{P}(q) + o(q)$. This yields $(\underline{\psi}_i, \overline{\psi}_i) = (\underline{\vartheta}_i, \overline{\vartheta}_i)$ for $i = 1, 2, 3$.

In general, Proposition 5.8 implies that $\mathrm{L}_1(q)/q = \mathrm{P}_1(q)/q + o(1)$, from which we deduce $(\underline{\psi}_1, \overline{\psi}_1) = (\underline{\vartheta}_1, \overline{\vartheta}_1)$. As i tends to infinity, we also have

$$\sup_{q\in I_i'} \frac{\mathrm{P}_3(q)}{q} = \frac{\mathrm{P}_3(c_i)}{c_i} \le \frac{1}{2-\delta+\sigma} + o(1) \quad \text{and} \quad \sup_{q\in I_i} \frac{\mathrm{P}_3(q)}{q} = \frac{\mathrm{P}_3(q_i)}{q_i} = \frac{1-\delta}{2-\delta} + o(1), \tag{51}$$

and the upper bound for $\overline{\psi}_3$ follows easily, since $\mathrm{L}_3(q) \le \mathrm{P}_3(q) + o(q)$. From now on we focus solely on the exponent $\widehat{\lambda}_{\min}(\xi)$. Suppose that $\delta < h(\sigma)$, or equivalently that $1/(2-\delta+\sigma) < (1-\delta)/(2-\delta)$. Note that $\mathbf{s}$ is bounded, since otherwise $\delta = \sigma = 0$. Fix ψ with

$$\frac{1}{2-\delta+\sigma} < \psi < \overline{\psi}_3 = \frac{1-\delta}{2-\delta}.$$

Let us prove that the exponent $\widehat{\lambda}_{\min}(\xi)$ can be computed by using only the points $(\mathbf{y}_i)_{i\geq 0}$. For each non-zero $\mathbf{y} \in \mathbb{Z}^3$, we denote by $q(\mathbf{y}) := \log\|\mathbf{y}\| - \log\|\mathbf{y}\wedge\Xi\|$ the abscissa at which $\mathrm{L}^*_{\mathbf{y}}$ changes slope. With this notation, we have $r_i := q(\mathbf{y}_i) = q_i + o(q_i)$. Eq. (45) yields

$$\frac{\mathrm{L}^*_{\mathbf{y}_i}(r_i)}{r_i} = -\frac{1-\delta}{2-\delta} + o(1),$$

from which we deduce that $\mathrm{L}^*_{\mathbf{y}_i}(r_i) \leq -\psi r_i$ for each large enough i. Conversely, let $\mathbf{y} \in \mathbb{Z}^3$ be a non-zero primitive point satisfying $\mathrm{L}^*_{\mathbf{y}}(q) \leq -\psi q$, where $q := q(\mathbf{y})$. Proposition 5.8 gives $\mathrm{L}^*_{\mathbf{y}}(q) \geq \mathrm{L}^*_1(q) = -\mathrm{L}_3(q) + \mathcal{O}(1) \geq -\mathrm{P}_3(q) + o(q)$. Combined with the left-hand side of (51) we obtain $q \notin \bigcup_{i>0} I'_i$ if $\|\mathbf{y}\|$ (and thus q) is large enough. Then, there is an index $i \geq 0$ such that $q \in I_i$. Suppose now that $\mathbf{y}$ and $\mathbf{y}_i$ are linearly independent. Then, since $\mathrm{L}^*_1(q) = \mathrm{L}^*_{\mathbf{y}_i}(q) + o(q)$, we get $\mathrm{L}^*_2(q) \leq \mathrm{L}^*_{\mathbf{y}}(q) + o(q)$, hence

$$\mathrm{L}_2(q) = -\mathrm{L}^*_2(q) + \mathcal{O}(1) \geq -\mathrm{L}^*_{\mathbf{y}}(q) + o(q) \geq \psi q + o(q).$$

Yet, $\psi > 1/(2+\sigma) = \overline{\psi}_2$, so, if q is large enough, $\mathbf{y}$ is proportional to $\mathbf{y}_i$. Finally, note that $\mathrm{L}^*_{\mathbf{y}}(q) \leq -\psi q$ is equivalent to $\|\mathbf{y}\wedge\Xi\| \leq \|\mathbf{y}\|^{-\mu}$, where $\mu = \psi/(1-\psi)$. By the above, the sequence of primitive points $\mathbf{y} \in \mathbb{Z}^3$ such that $\|\mathbf{y}\wedge\Xi\| \leq \|\mathbf{y}\|^{-\mu}$ coincides, up to a finite numbers of terms, with the sequence $(\mathbf{y}_i)_{i\geq 0}$. We deduce by a classical reasoning (see for example [25, Section 2]) that

$$\widehat{\lambda}_\mu(1,\xi,\xi^2) = \liminf_{i\to\infty} -\frac{\log\|\mathbf{y}_i\wedge\Xi\|}{\log\|\mathbf{y}_{i+1}\|} = \liminf_{i\to\infty} -\frac{\log\Delta^*_i}{\log Y_{i+1}}.$$

Let us write $i = t_k + \ell$, with $k \geq 0$ and $0 \leq \ell < s_{k+1}$. Since the quotient

$$-\frac{\log\Delta^*_i}{\log Y_{i+1}} = \frac{(1-\delta)\big((\ell+1)\log W_k + \log W_{k-1}\big)}{(\ell+2)\log W_k + \log W_{k-1}}$$

is increasing with ℓ, it is minimum for $\ell = 0$, and we get by (42)

$$\widehat{\lambda}_\mu(1,\xi,\xi^2) = \liminf_{k\to\infty} \frac{(1-\delta)\big(\log W_k + \log W_{k-1}\big)}{2\log W_k + \log W_{k-1}} = \frac{(1-\delta)(1+\sigma)}{2+\sigma}.$$

We deduce the value of $\widehat{\lambda}_{\min}(\xi)$ from the above by noticing that $\mu = \psi/(1-\psi)$ tends to $\overline{\psi}_3/(1-\overline{\psi}_3) = \lambda_2(\xi)$ as ψ tends to $\overline{\psi}_3$. $\square$

Theorem 5.9 combined with (39) implies the following result.

Corollary 5.10. *Let $\xi \in \mathit{Sturm}(\mathbf{s})$ and let $\delta = \delta(\xi) \in [0, \sigma/(1+\sigma)]$ denote the quantity associated to ξ as in Theorem 5.9. Then*

$$\widehat{\omega}_2(\xi) = 1 + (1-\delta)(1+\sigma), \qquad \widehat{\lambda}_2(\xi) = \frac{(1-\delta)(1+\sigma)}{1+(1-\delta)(1+\sigma)},$$
$$\omega_2(\xi) = \frac{2-\delta}{\sigma} + 1 - \delta, \qquad 1-\delta \leq \lambda_2(\xi) \leq \max\Big(1-\delta, \frac{1}{1-\delta+\sigma}\Big) \leq 1.$$

If moreover δ satisfies the stronger condition $\delta < h(\sigma)$, where $h(\sigma) = \sigma/2 + 1 - \sqrt{(\sigma/2)^2+1}$, then

$$\lambda_2(\xi) = 1-\delta \quad \text{and} \quad \widehat{\lambda}_{\min}(\xi) = \frac{(1-\delta)(1+\sigma)}{2+\sigma}.$$

The formula for $\lambda_2(\xi)$ still holds if $\delta = h(\sigma)$.

5.3. **Approximation by algebraic numbers of degree at most** 2.

In view of Corollary 5.10, the proof of Theorem 1.3 essentially reduces to showing $\omega_2(\xi) = \omega^*_2(\xi)$ and $\widehat{\omega}_2(\xi) = \widehat{\omega}^*_2(\xi)$ for the numbers $\xi \in \mathit{Sturm}(\mathbf{s})$. Following a suggestion of Yann Bugeaud, we prove the following slightly stronger statement.

Proposition 5.11. *Let $\xi \in Sturm(\mathbf{s})$. Then*

$$\omega^*_{2,exact}(\xi) = \omega^*_2(\xi) = \omega_2(\xi) \quad and \quad \widehat{\omega}^*_{2,exact}(\xi) = \widehat{\omega}^*_2(\xi) = \widehat{\omega}_2(\xi),$$

where $\omega^*_{2,exact}(\xi)$ (resp. $\widehat{\omega}^*_{2,exact}(\xi)$) is the supremum of all real numbers τ such that, for arbitrarily large values of X (resp. for each X large enough), there is an algebraic number α of degree exactly 2 satisfying

$$H(\alpha) \le X \quad and \quad |\xi - \alpha| \le H(\alpha)^{-1}X^{-\tau}.$$

The problem of approximating a real number by algebraic numbers of prescribed degree has been studied by several authors, see for example [12], [33] and [34].

Before proving Proposition 5.11, let us explain our strategy. First, by (9) we have the general estimates

$$\omega^*_{2,\mathrm{exact}}(\xi) \le \omega^*_2(\xi) \le \omega_2(\xi) \quad \text{and} \quad \widehat{\omega}^*_{2,\mathrm{exact}}(\xi) \le \widehat{\omega}^*_2(\xi) \le \widehat{\omega}_2(\xi). \tag{52}$$

By definition of $\omega^*_{2,\mathrm{exact}}(\xi)$ and $\widehat{\omega}^*_{2,\mathrm{exact}}(\xi)$, the reverse inequalities of (52) hold if the best solutions $P \in \mathbb{Z}^3 \cong \mathbb{R}[X]_{\le 2}$ of the problems defining $\omega_2(\xi)$ and $\widehat{\omega}_2(\xi)$ (see Section 2), have two quadratic roots α, α', with $|\alpha' - \xi| \asymp 1$. Indeed, in that case we have $|\alpha - \xi| \asymp |P(\xi)|/H(\alpha)$. Here, Proposition 5.8 indicates that the relevant polynomials to be considered correspond to the points $\mathbf{z}_{t_k}$ of Section 5.2. We will also show that those polynomials are irreducible if k is large enough.

For each $\xi \in \mathbb{R}$ we define $\lambda_1(\xi)$ as the supremum of real numbers λ such that, for arbitrarily large values of X there exists $(p,q) \in \mathbb{Z}^2 \setminus \{0\}$ satisfying $|q\xi - p| \le X^{-\lambda}$ and $|q| \le X$. By a general transference inequality due to Bugeaud [8, Lemma 3.1], we have $\lambda_1(\xi) \le 2\lambda_2(\xi)+1$ for each ξ which is neither rational nor quadratic. Combined with Corollary 5.10, we obtain the following result.

Lemma 5.12. *Let $\xi \in Sturm(\mathbf{s})$. Then $\lambda_1(\xi) \le 3$.*

Remark. The upper bound $\lambda_1(\xi) \le 3$ could be greatly improved by generalizing the arguments of [28, §7]. It seems that $\lambda_1(\xi) = 1$ when the parameter $\delta(\xi)$ is equal to 0. It would be interesting to know if there exists $\xi \in Sturm(\mathbf{s})$ satisfying $\lambda_1(\xi) > 1$.

Lemma 5.13. *Let $(\mathrm{w}_k)_{k\ge 0}$ be an admissible ψ-Sturmian sequence with multiplicative growth. We denote by $(\mathbf{b}_i)_{i\ge -1}$ and $(\mathbf{a}_i)_{i\ge -1}$ the sequences of symmetric matrices associated by Definition 3.10. We suppose that $(\mathbf{a}_i)_{i\ge -1}$ converges projectively to a symmetric matrix M_ξ identified with $(1,\xi,\xi^2)$, where ξ neither rational nor quadratic. Then, there exist two distinct non-zero real numbers $\xi', \xi'' \in \mathbb{R} \setminus \{\pm\xi\}$ with the following properties. The point $\mathbf{b}_i$ converges projectively to $P_0 = (X-\xi)(X-\xi')$ (resp. $P_1 = (X-\xi)(X-\xi'')$) as $i = t_k + \ell$ tends to infinity with $k \ge 0$ even (resp. odd) and $0 \le \ell < s_{k+1}$.*

Proof. We write $\Xi = (1,\xi,\xi^2)$ and denote by N the matrix $(\mathrm{Id} - \mathrm{w}_1^{-1}\mathrm{w}_0^{-1}\mathrm{w}_1\mathrm{w}_0)J$. Proposition 3.11 gives, for any k, ℓ with $k \ge 1$ and $0 \le \ell < s_{k+1}$,

$$\mathbf{b}_{t_k+\ell} = U(\mathrm{w}_k^\ell \mathrm{w}_{k-1}) = U(\mathbf{a}_{\psi(t_k+\ell)} N_k^{-1})$$

(see (16) for the case $\ell = 0$). We deduce that $\mathbf{b}_{t_k+\ell}$ converges projectively to $Q_0 := U(M_\xi N^{-1})$ (resp. $Q_1 := U(M_\xi {}^tN^{-1})$) as $i = t_k+\ell$ tends to infinity with k even (resp. k odd) and $0 \le \ell < s_{k+1}$. Now, let us identify Q_0, Q_1 and $\mathbf{b}_i$ with their corresponding polynomial of degree ≤ 2 as in Section 2. Explicitly, we have the formulas

$$\left\{ \begin{array}{ll} Q_0 & = -\xi(a+c\xi) + (a+(c-b)\xi - d\xi^2)X + (b+d\xi)X^2 \\ Q_1 & = -\xi(a+b\xi) + (a+(b-c)\xi - d\xi^2)X + (c+d\xi)X^2 \end{array} \right. , \quad \text{where } N^{-1} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

Recall that N is invertible, neither symmetric nor antisymmetric by Lemma 4.1, and ξ is not the root of a polynomial in $\mathbb{Z}[X]$ of degree 2. This implies that none of the coefficients of Q_0 and Q_1

is zero. By the above, the discriminant Δ of Q_0 and Q_1 is equal to $\Delta = (a+(b+c)\xi+d\xi^2)^2 > 0$. The two distincts roots of Q_0 (resp. Q_1) are ξ and

$$\xi' := -\frac{a+c\xi}{b+d\xi} \quad \Big(\text{resp. } \xi'' := -\frac{a+b\xi}{c+d\xi}\Big).$$

and $\xi' \neq \xi''$ (since N is neither symmetric nor antisymmetric) and $\xi', \xi'' \notin \{0, -\xi\}$ since Q_0 and Q_1 have non-zero coefficients. □

Proof of Proposition 5.11. We write $\Xi = (1, \xi, \xi^2)$ and we keep the notation of Section 5.2 for the ψ-Sturmian sequence $(\mathrm{w}_k)_{k\geq 0}$, the parameter $\delta \in [0, \sigma/(1+\sigma)]$ and the sequence of primitive integer points $(\mathbf{z}_i)_{i\geq -1}$ (also viewed as a sequence of polynomials) associated with $(\mathbf{b}_i)_{i\geq -1}$ as in Theorem 4.11. Let P_0, P_1 and ξ', ξ'' be as in Lemma 5.13. Since P_0 and P_1 have non-zero coefficients and positive discriminant, each of the coordinates of $\mathbf{z}_i$ is $\asymp \|\mathbf{z}_i\|$, and the discriminant of $\mathbf{z}_i$ is positive for large enough i. For those i, we denote by r_i and r_i' the two real roots of $\mathbf{z}_i$, with $|r_i - \xi| \leq |r_i' - \xi|$. The root r_i' converges to ξ' (resp. ξ'') as $i = t_k + \ell$ tends to infinity with k even (resp. odd) and $0 \leq \ell < s_{k+1}$. Since $\xi', \xi'' \neq \xi$, we have $|\xi - r_i'| \asymp 1$. This yields

$$H(r_i) \ll \|\mathbf{z}_i\| \quad \text{and} \quad |\xi - r_i| \asymp \frac{|\mathbf{z}_i\cdot\Xi|}{\|\mathbf{z}_i\|} \ll \frac{|\mathbf{z}_i\cdot\Xi|}{H(r_i)}. \tag{53}$$

Note that $H(r_i)$ tends to infinity since r_i converges to ξ which is neither rational nor quadratic. For each $k \geq 1$ let $\varepsilon_k = [s_k; s_{k-1}, \dots, s_1] \geq \gamma - o(1)$, so that $W_k = W_{k-1}^{\varepsilon_k}$, where W_k is defined as in Section 5.2 (see (41), recall that $\gamma = 1.618\cdots$ denotes the golden ratio). For $i := t_k$, we find $H(r_{t_k}) \leq W_{k-1}^{1+o(1)}$ and (45) leads us to

$$|\mathbf{z}_{t_k}\cdot\Xi| = W_k^{-(2-\delta+o(1))}W_{k-1}^{-(1-\delta)} = W_{k-1}^{-((2-\delta+o(1))\varepsilon_k+1-\delta)} \leq H(r_{t_k})^{-((2-\delta+o(1))\varepsilon_k+1-\delta)}.$$

The above combined with (53) gives

$$H(r_{t_k})|\xi - r_{t_k}| \ll H(r_{t_k})^{-((2-\delta+o(1))\varepsilon_k+1-\delta)}. \tag{54}$$

Now, note that $\sigma/(1+\sigma)$ is increasing with $\sigma \leq 1/\gamma$, and thus $\delta \leq 1/\gamma^2$. We deduce that

$$(2-\delta+o(1))\varepsilon_k + 1 - \delta \geq (2-\delta)\gamma + 1 - \delta + o(1) = 2\gamma + 1 - \gamma^2\delta \geq 2\gamma + o(1).$$

Since $2\gamma > 3$ and $\lambda_1(\xi) \leq 3$ by Lemma 5.12, Eq. (54) implies that $r_{t_k} \notin \mathbb{Q}$ for each k large enough. Consequently, for each large k, the polynomial z_{t_k} is irreducible over $\mathbb{Q}$ and of degree 2, and therefore $H(r_{t_k}) = \|\mathbf{z}_{t_k}\|$.

Since $\limsup_{k\to\infty}\varepsilon_k = 1/\sigma$ by (42), in view of (54) and Corollary 5.10, we get

$$\omega_{2,\mathrm{exact}}^*(\xi) \geq \limsup_{k\to\infty}\big((2-\delta+o(1))\varepsilon_k + 1 - \delta\big) = \frac{2-\delta}{\sigma} + 1 - \delta = \omega_2(\xi).$$

Similarly, let X be a large real number, and let k be such that $\|\mathbf{z}_k\| \leq X < \|\mathbf{z}_{k+1}\| = W_k^{1+o(1)}$. If k is large enough, then $\|\mathbf{z}_{t_k}\| = H(r_{t_k}) \leq X$. We obtain

$$|\mathbf{z}_{t_k}\cdot\Xi| = W_k^{-(2-\delta+o(1))}W_{k-1}^{-(1-\delta)} = W_k^{-(2-\delta+o(1)+\varepsilon_k^{-1}(1-\delta))} \leq X^{-(2-\delta+o(1)+\varepsilon_k^{-1}(1-\delta))}.$$

Using the above with (53), we obtain

$$\widehat{\omega}_{2,\mathrm{exact}}^*(\xi) \geq \liminf_{k\to\infty}\big(2-\delta+o(1)+\varepsilon_k^{-1}(1-\delta)\big) = 2 - \delta + (1-\delta)\sigma = \widehat{\omega}_2(\xi).$$

□

5.4. **Proofs.**

Proof of Theorem 1.3. As seen in the introduction, $\mathit{Sturm}(\mathbf{s})$ is as most countable, and the density of $\Delta(\mathbf{s})$ in $[0,\sigma/(1+\sigma)]$ when $\sigma>0$ (or equivalently $\mathbf{s}$ bounded) comes from the construction of ψ-Sturmian numbers (see [24, Section 9]). The part concerning the exponents is a consequence of Corollary 5.10 and Proposition 5.11. Since Bugeaud-Laurent continued fractions ξ_φ belong to $\mathit{Sturm}(\mathbf{s})$ and $\delta(\xi_\varphi)=0$, we have $0\in\Delta_{\mathbf{s}}$. □

Corollary 1.5 is a consequence of the following result. As defined in the introduction, $\mathbf{1}=(s_k)_{k\geq 1}$ is the constant sequence $s_k=1$ for each $k\geq 1$.

Lemma 5.14. *There exists $\varepsilon>0$ with the following property. For each $\xi\in\mathbb{R}$ which is neither rational nor quadratic, if $\widehat{\omega}_2(\xi)\geq\gamma^2-\varepsilon$, then $\xi\in\mathrm{Sturm}(\mathbf{1})$.*

Proof. The set of points $(1,\eta,\eta^2)$ with $\eta\in\mathbb{R}$ corresponds to a quadratic hypersurface associated to the quadratic form $x_0x_2-x_1^2$. As a consequence of [26] (see [26, Theorem 7.3]), for each η with $0<\eta<1$, there is $\varepsilon'>0$ with the following property. Let $\xi\in\mathbb{R}$ (which is neither rational nor quadratic) with $\widehat{\lambda}_2(\xi)\geq 1/\gamma-\varepsilon'$, and write $\Xi=(1,\xi,\xi^2)$. Then, there exists a sequence of primitive points $(\mathbf{v}_i)_{i\geq 0}$ such that

(i) The sequence $(\|\mathbf{v}_i\|)_{i\geq 0}$ tends to infinity.

(ii) The matrix $\mathbf{v}_{i+1}$ is proportional to $\mathbf{v}_i\mathrm{Adj}(\mathbf{v}_{i-2})\mathbf{v}_i$ for each large enough i.

(iii) We have $\|\mathbf{v}_i\wedge\Xi\|\ll\|\mathbf{v}_i\|^{-1+\eta}$.

The above phenomenon was first observed by Fischler in an unpublished work. Using the classical estimate $|\det(\mathbf{v}_i)|\ll\|\mathbf{v}_i\|\|\mathbf{v}_i\wedge\Xi\|$, we deduce from (iii) that $\limsup_{i\to\infty}\log|\det(\mathbf{v}_i)|/\log\|\mathbf{v}_i\|$ is at most η. So, if we choose $\eta\leq\sigma/(1+\sigma)$ (where $\sigma=\sigma(\mathbf{1})=1/\gamma$), we find $\xi\in\mathit{Sturm}(\mathbf{1})$. By Jarník's identity (8), there exists $\varepsilon>0$ such that $\widehat{\omega}_2(\xi)\geq\gamma^2-\varepsilon$ implies $\widehat{\lambda}_2(\xi)\geq 1/\gamma-\varepsilon'$. □

Proof of Theorem 1.8. Let $\xi\in\mathbb{R}$ which is neither rational nor quadratic with $\beta_0(\xi)<\sqrt{3}$. Let $(\mathbf{v}_i)_{i\geq 0}$ and $\mathbf{s}$ be the sequences given by Theorem 1.8. Since $\mathbf{s}$ is bounded, we have $\sigma=\sigma(\mathbf{s})>0$. The first two conditions of Definition 1.9 are satisfied. The last one comes from the estimates $|\det(\mathbf{y}_i)|\ll\|\mathbf{v}_i\|\|\mathbf{v}_i\wedge\Xi\|=\|\mathbf{v}_i\|^{o(1)}$ as i tends to infinity, where $\Xi=(1,\xi,\xi^2)$. We therefore have $\xi\in\mathit{Sturm}(\mathbf{s})$ and by the above, the parameter $\delta(\xi)$ (see (50)) is equal to 0, and thus $<h(\sigma)$. Theorem 1.3 yields $\widehat{\lambda}_2(\xi)=\widehat{\lambda}_{\min}(\xi)$. We conclude by recalling that $\beta_0(\xi)<2$ implies that $\widehat{\lambda}_{\min}(\xi)=1/\beta_0(\xi)$ (see Section 2). □

Proof of Definition 1.2 $\Leftrightarrow$ Definition 1.9. The implication $\Leftarrow$ follows from Theorem 4.11 and the estimates (45).
$\Rightarrow$ Let ξ and $(\mathrm{w}_k)_{k\geq 0}$ be as in Definition 1.2. The first two conditions ensure that $(\mathrm{w}_k)_{k\geq 0}$ is admissible and has multiplicative growth. Note that $(\widetilde{\mathrm{w}}_k)_{k\geq 0}$ (and thus $(\mathrm{w}_k)_{k\geq 0}$) is unbounded since $P_k(\xi)$ tends to 0 and ξ is neither rational nor quadratic. By Proposition 4.6, there are $\alpha,\beta,\varrho\geq 0$, with $\beta>0$ and $\alpha\leq 2\beta$, such that $\|\mathrm{w}_k\|\asymp e^{\beta p_k}$, $|\det(\mathrm{w}_k)|\asymp e^{\alpha p_k}$ and $c_k=e^{p_k(\varrho+o(1))}$. The condition (iv) can be rewritten as $(\alpha-2\varrho)/(\beta-\varrho)\leq\sigma/(1+\sigma)$. In particular, we must have $\alpha/\beta<2$, and therefore Proposition 4.5 applies. We obtain that the sequence of symmetric matrices $(\mathbf{y}_i)_{i\geq -2}$ associated to $(\widetilde{\mathrm{w}}_k)_{k\geq 0}$ converges projectively to a point $(1,\eta,\eta^2)$, which, by Lemma 5.13 combined with condition (iii) of Definition 1.2, is equal to $(1,\xi,\xi^2)$. Then, the sequence of primitive integer points $(\mathrm{cont}(\mathbf{y}_i)^{-1}\mathbf{y}_i)_{i\geq 0}$ is as in Definition 1.9. □

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