PARAMETRIC GEOMETRY OF NUMBERS OVER A NUMBER FIELD AND EXTENSION OF SCALARS

ANTHONY POËLS AND DAMIEN ROY

Abstract. Schmidt and Summerer parametric geometry of numbers deals with rational approximation to points in $\mathbb{R}^n$. We extend this theory to a number field K and its completion K_w at a place w in order to treat approximation over K to points in K_w^n. As a consequence, we find that exponents of approximation over $\mathbb{Q}$ in $\mathbb{R}^n$ have the same spectrum as their generalizations over K in K_w^n. When w has relative degree one over a place ℓ of $\mathbb{Q}$, we further relate approximation over K to a point $\boldsymbol{\xi}$ in K_w^n, to approximation over $\mathbb{Q}$ to a point Ξ in $\mathbb{Q}_\ell^{nd}$, obtained by extension of scalars, where d is the degree of K over $\mathbb{Q}$. By combination with a result of P. Bel, this allows us to construct algebraic curves in $\mathbb{R}^{3d}$ defined over $\mathbb{Q}$, of degree $2d$, containing points that are very singular with respect to rational approximation.

1. Introduction

In Diophantine approximation, one is interested in measuring how well a given non-zero point $\boldsymbol{\xi} \in \mathbb{R}^n$ with $n \geq 2$ can be approximated by subspaces of $\mathbb{R}^n$ defined over $\mathbb{Q}$ of a given dimension k. The most important cases are $k = 1$ and $k = n-1$, and each gives rise naturally to a pair of exponents of approximation. For $k = 1$, they are $\widehat{\lambda}(\boldsymbol{\xi})$ (resp. $\lambda(\boldsymbol{\xi})$) defined as the supremum of all real numbers λ for which the inequalities

$$\|\mathbf{x}\| \leq Q \quad \text{and} \quad \|\mathbf{x} \wedge \boldsymbol{\xi}\| \leq Q^{-\lambda} \tag{1.1}$$

have a non-zero solution $\mathbf{x} \in \mathbb{Z}^n$ for each large enough $Q \geq 1$ (resp. for arbitrarily large values of $Q \geq 1$). For $k = n-1$, they are $\widehat{\omega}(\boldsymbol{\xi})$ (resp. $\omega(\boldsymbol{\xi})$) defined as the supremum of all real numbers ω for which the inequalities

$$\|\mathbf{x}\| \leq Q \quad \text{and} \quad |\mathbf{x} \cdot \boldsymbol{\xi}| \leq Q^{-\omega} \tag{1.2}$$

have a non-zero solution $\mathbf{x} \in \mathbb{Z}^n$ for each large enough $Q \geq 1$ (resp. for arbitrarily large values of $Q \geq 1$), where the dot represents the usual scalar product in $\mathbb{R}^n$. This is independent of the choice of norms in $\mathbb{R}^n$ and in $\bigwedge^2 \mathbb{R}^n$ but for convenience, we use the euclidean norms. As these exponents depend only on the class of $\boldsymbol{\xi}$ in $\mathbb{P}^{n-1}(\mathbb{R})$, we may assume that $\|\boldsymbol{\xi}\| = 1$. We refer the reader to the paper of Laurent [9] for generalizations in intermediate dimensions k.

In studying such exponents, it is important to restrict to points $\boldsymbol{\xi} \in \mathbb{R}^n$ with $\mathbb{Q}$-linearly independent coordinates, as this yields simpler statements and can be achieved by dropping

2020 *Mathematics Subject Classification.* Primary 11J13; Secondary 11J82.

Key words and phrases. Adelic geometry of numbers, Diophantine approximation, exponents of approximation, extension of scalars, extremal numbers, n-systems, number fields, parametric geometry of numbers, rational points, simultaneous approximation, spectrum of exponents, transference inequalities.

Work of both authors partially supported by NSERC.

redundant coordinates if necessary. For such points, a result of Dirichlet gives

$$(n-1)^{-1} \le \widehat{\lambda}(\boldsymbol{\xi}) \le \lambda(\boldsymbol{\xi}) \quad \text{and} \quad n-1 \le \widehat{\omega}(\boldsymbol{\xi}) \le \omega(\boldsymbol{\xi}).$$

However, this does not fully describe the *spectrum* of $(\widehat{\lambda}, \lambda, \widehat{\omega}, \omega)$, namely the set of all quadruples $(\widehat{\lambda}(\boldsymbol{\xi}), \lambda(\boldsymbol{\xi}), \widehat{\omega}(\boldsymbol{\xi}), \omega(\boldsymbol{\xi}))$ associated with these $\boldsymbol{\xi}$. For $n = 2$, a complete description is simply given by

$$1 = \widehat{\lambda}(\boldsymbol{\xi}) = \widehat{\omega}(\boldsymbol{\xi}) \le \lambda(\boldsymbol{\xi}) = \omega(\boldsymbol{\xi}) \le \infty$$

For $n = 3$, the description is more complicated and was achieved by Laurent in [8], showing it as a semi-algebraic set. One of the constraints that it involves is the following remarkable identity due to Jarník [7, Satz 1],

$$\frac{1}{\widehat{\lambda}(\boldsymbol{\xi})} - 1 = \frac{1}{\widehat{\omega}(\boldsymbol{\xi}) - 1}, \tag{1.3}$$

which together with $2 \le \widehat{\omega}(\boldsymbol{\xi}) \le \infty$ fully describes the spectrum of the pair $(\widehat{\lambda}, \widehat{\omega})$. For $n \ge 4$, the spectrum of the four exponents is not known but Marnat [11] has shown that it contains an open subset of $\mathbb{R}^4$ and thus it obeys no algebraic relation such as (1.3).

Many of the recent progresses, including the breakthrough of Marnat and Moshchevitin [12] who determined the spectra of the pairs $(\widehat{\lambda}, \lambda)$ and $(\widehat{\omega}, \omega)$ for each $n \ge 3$, use in a crucial way Schmidt's and Summerer's parametric geometry of numbers [22]. In the dual but equivalent setting of [17], this theory attaches to any point $\boldsymbol{\xi} \in \mathbb{R}^n$ with $\|\boldsymbol{\xi}\| = 1$, the family of symmetric convex bodies of $\mathbb{R}^n$

$$\mathcal{C}_{\boldsymbol{\xi}}(q) = \{\mathbf{x} \in \mathbb{R}^n\,;\, \|\mathbf{x}\| \le 1 \text{ and } |\mathbf{x} \cdot \boldsymbol{\xi}| \le e^{-q}\} \subseteq \mathbb{R}^n$$

parametrized by real numbers $q \ge 0$. For each $j = 1, \dots, n$, let $L_{\boldsymbol{\xi},j}(q)$ denote the logarithm of the j-th minimum of $\mathcal{C}_{\boldsymbol{\xi}}(q)$ with respect to $\mathbb{Z}^n$, namely the smallest real number t such that $e^t\mathcal{C}_{\boldsymbol{\xi}}(q)$ contains at least j linearly independent points of $\mathbb{Z}^n$. Then, form the map

$$\begin{array}{rccc} \mathbf{L}_{\boldsymbol{\xi}} \colon & [0,\infty) & \longrightarrow & \mathbb{R}^n \\ & q & \longmapsto & (L_{\boldsymbol{\xi},1}(q), \dots, L_{\boldsymbol{\xi},n}(q))\,. \end{array} \tag{1.4}$$

Translated in this setting, Schmidt and Summerer first observe in [22] that the standard exponents of approximation to $\boldsymbol{\xi}$, including the four ones mentioned above, are given by simple formulas in terms of the inferior and superior limits of the ratios $L_{\boldsymbol{\xi},j}(q)/q$ as q goes to infinity. Secondly they show the existence of a constant $\gamma \ge 0$ and of a continuous piecewise linear map $\mathbf{P} \colon [0,\infty) \to \mathbb{R}^n$ with growth conditions involving γ, such that the difference $\mathbf{L}_{\boldsymbol{\xi}} - \mathbf{P}$ is bounded. Thus the above mentioned exponents of approximation to $\boldsymbol{\xi}$ can be computed, via the same formulas, in terms of the behaviour of $\mathbf{P}$ at infinity. They call such a map $\mathbf{P}$ an (n,γ)-system, and their set increases as the deformation parameter γ increases. The $(n,0)$-systems, whose simpler description is recalled in Section 2, are simply called n-systems for shortness.

The main result of [17] provides a converse and shows more precisely that the set of maps $\mathbf{L}_{\boldsymbol{\xi}}$ with $\boldsymbol{\xi} \in \mathbb{R}^n$ and $\|\boldsymbol{\xi}\| = 1$ coincides with the set of n-systems modulo the additive group of bounded functions from $[0,\infty)$ to $\mathbb{R}^n$. Moreover, $\boldsymbol{\xi}$ has $\mathbb{Q}$-linearly independent coordinates if and only if any corresponding n-system $\mathbf{P} = (P_1, \dots, P_n)$ satisfies $\lim_{q\to\infty} P_1(q) = \infty$. This reduces the determination of the spectra of a family of exponents of approximation to a combinatorial problem about such n-systems.

A similar theory is developped in [19], with $\mathbb{Q}$ replaced by a field of rational functions in one variable $F(T)$ over an arbitrary field F, and $\mathbb{R}$ replaced by the completion $F((1/T))$ of $F(T)$ for the degree valuation.

The first goal of this paper is to extend the theory to a number field K and its completion K_w at a place w, in order to study approximation over K to an arbitrary non-zero point $\boldsymbol{\xi}$ of K_w^n. In the next section we show how to attach to such a point a function $\mathbf{L}_{\boldsymbol{\xi}}\colon [0,\infty) \to \mathbb{R}^n$ from which the four exponents of approximation to $\boldsymbol{\xi}$ can be computed in the same way as in the case where K is $\mathbb{Q}$ and w is its place at infinity. We will show that this set of maps also coincides with the set of n-systems modulo bounded functions. Thus the spectrum of these exponents remains the same in this new context. In particular, Jarník's identity (1.3) holds for any point $\boldsymbol{\xi}$ of K_w^3 with linearly independent coordinates over K.

The second goal of this paper deals with extension of scalars from $\mathbb{Q}$ to a number field K. For this we assume that w is a place of K of relative degree one over $\mathbb{Q}$, so that $K_w = \mathbb{Q}_\ell$ for the place ℓ of $\mathbb{Q}$ induced by w. We also choose a basis $\boldsymbol{\alpha} = (\alpha_1, \dots, \alpha_d)$ of K over $\mathbb{Q}$ and for each point $\boldsymbol{\xi} = (\xi_1, \dots, \xi_n) \in K_w^n$, we define

$$\Xi = \boldsymbol{\alpha} \otimes \boldsymbol{\xi} = (\alpha_1 \boldsymbol{\xi}, \dots, \alpha_d \boldsymbol{\xi}) \in K_w^{nd} = \mathbb{Q}_\ell^{nd} \tag{1.5}$$

and say that Ξ is obtained from $\boldsymbol{\xi}$ by *extending scalars from* $\mathbb{Q}$ *to* K. If $\boldsymbol{\xi}$ has linearly independent coordinates over K, then Ξ has linearly independent coordinates over $\mathbb{Q}$ and we will show a close relationship between the maps $\mathbf{L}_{\boldsymbol{\xi}}$ and $\mathbf{L}_\Xi$. From this we will deduce formulas linking the Diophantine exponents of approximation to $\boldsymbol{\xi}$ over K with those of Ξ over $\mathbb{Q}$. As a consequence, we will see that Jarník's identity (1.3) yields

$$\frac{1}{\widehat{\lambda}(\Xi)} - (2d-1) = \frac{d^2}{\widehat{\omega}(\Xi) - (2d-1)} \tag{1.6}$$

for any $\Xi = \boldsymbol{\alpha} \otimes \boldsymbol{\xi} \in \mathbb{Q}_\ell^{3d}$ constructed from a point $\boldsymbol{\xi} \in K_w^3$ with K-linearly independent coordinates.

Let ℓ be a place of $\mathbb{Q}$. We say that a point $\boldsymbol{\xi} \in \mathbb{Q}_\ell^n$ is *very singular* if it has linearly independent coordinates over $\mathbb{Q}$ and satisfies $\widehat{\lambda}(\boldsymbol{\xi}) > 1/(n-1)$. This requires $n \geq 3$. Moreover, by Schmidt's subspace theorem, such a point is not algebraic and so generates a field $\mathbb{Q}(\boldsymbol{\xi})$ of transcendence degree at least one over $\mathbb{Q}$. The third goal and the initial motivation of this paper is to provide new examples of very singular points of transcendence degree one. Up to now, all known examples come from dimension $n = 3$ and, aside from the constructions of [16], they are all of the form $\boldsymbol{\xi} = (1, \xi, \xi^2)$. Moreover, the supremum of $\widehat{\lambda}(1, \xi, \xi^2)$ for a transcendental number $\xi \in \mathbb{Q}_\ell$ is $1/\gamma \simeq 0.618$ where $\gamma = (1+\sqrt{5})/2$ denotes the Golden ratio. For $\mathbb{Q}_\ell = \mathbb{R}$, this follows from the constructions of [14] or [15] together with the upper bound of [6, Theorem 1a]. For a prime number ℓ, this follows from [25, Chapter 2] or [3] together with [23, Théorème 2]. More generally, Bel showed in [1] that the result extends to any number field K and its completion K_w at a place w. Assuming that w extends ℓ with relative degree 1 and choosing a basis $\boldsymbol{\alpha} = (\alpha_1, \dots, \alpha_d)$ of K over $\mathbb{Q}$, we will deduce that $\mathbb{Q}_\ell^{3d}$ contains very singular points of the form $(\boldsymbol{\alpha}, \xi\boldsymbol{\alpha}, \xi^2\boldsymbol{\alpha})$ with $\xi \in \mathbb{Q}_\ell$.

2. Notation and main results

Throughout this paper, we fix an algebraic extension K of $\mathbb{Q}$ of finite degree d.

2.1. **Absolute values.** We denote by $M(K)$ the set of non-trivial places of K, and by $M_\infty(K)$ the subset of its archimedean places. For each $v \in M(K)$, we denote by K_v the completion of K at v and by $d_v = [K_v : \mathbb{Q}_v]$ its local degree. When $v \in M_\infty(K)$, we normalize the absolute value $|\ |_v$ on K_v so that it extends the usual absolute value $|\ |_\infty$ on $\mathbb{Q}$. Then K_v embeds isometrically into $\mathbb{C}$. We identify it with its image $\mathbb{R}$ or $\mathbb{C}$, and write $v \mid \infty$. Otherwise, there is a unique prime number p with $|p|_v < 1$ and we ask that $|p|_v = p^{-1}$ so that $|\ |_v$ extends the usual p-adic absolute on $\mathbb{Q}$. We then write $v \mid p$. For these normalizations and each $a \in K^*$, the product formula reads

$$\prod_{v \in M(K)} |a|_v^{d_v/d} = 1.$$

2.2. **Local norms and heights.** Given a positive integer n and a place $v \in M(K)$, we define the norm of a point $\mathbf{x} = (x_1, \dots, x_n)$ in K_v^n by

$$\|\mathbf{x}\|_v = \begin{cases} (|x_1|_v^2 + \cdots + |x_n|_v^2)^{1/2} & \text{if } v \mid \infty, \\ \max\{|x_1|_v, \dots, |x_n|_v\} & \text{otherwise.} \end{cases}$$

For this choice of local norms, we define the *height* of a non-zero point $\mathbf{x}$ in K^n by

$$H(\mathbf{x}) = \prod_{v \in M(K)} \|\mathbf{x}\|_v^{d_v/d}.$$

By the product formula, it depends only on the class of $\mathbf{x}$ in $\mathbb{P}^{n-1}(K)$ and satisfies $H(\mathbf{x}) \geq 1$.

More generally, for each $k \in \{1, \dots, n\}$ and each $v \in M(K)$, we define the norm of a point in $\bigwedge^k K_v^n$ to be the norm of its set of Plücker coordinates in K_v^N where $N = \binom{n}{k}$. We also define the height of a point in $\bigwedge^k K^n$ to be the height of its set of Plücker coordinates in K^N. This is independent of the ordering of these coordinates. Then, we define the height of a k-dimensional subspace V of K^n to be

$$H(V) = H(\mathbf{x}_1 \wedge \cdots \wedge \mathbf{x}_k)$$

independently of the choice of a basis $(\mathbf{x}_1, \dots, \mathbf{x}_k)$ of V over K. For the subspace 0 of K^n, we set $H(0) = 1$.

2.3. **The canonical bilinear form.** We endow K^n with the bilinear form given by

$$\mathbf{x}\cdot\mathbf{y} = x_1y_1 + \cdots + x_ny_n \tag{2.1}$$

for each $\mathbf{x} = (x_1, \dots, x_n)$ and $\mathbf{y} = (y_1, \dots, y_n)$ in K^n. Then we define the orthogonal space to a subspace V of K^n to be

$$V^\perp = \{\mathbf{y} \in K^n\,;\, \mathbf{x}\cdot\mathbf{y} = 0 \text{ for each } \mathbf{x} \in V\}. \tag{2.2}$$

According to a result of Schmidt, it has the same height $H(V^\perp) = H(V)$ as V.

For each $v \in M(K)$, the same formula (2.1) provides a bilinear form on K_v^n which we denote in the same way. For a subspace V of K_v^n, we also define $V^\perp$ by (2.2) but allowing $\mathbf{y}$ to run through K_v^n.

2.4. **Exponents of approximation.** Fix a place $w \in M(K)$ and a non-zero point $\boldsymbol{\xi} \in K_w^n$. For each non-zero $\mathbf{x} \in K^n$, we modify slightly the notation of P. Bel in [1] by setting

$$D^*_{\boldsymbol{\xi}}(\mathbf{x}) = \Big(\frac{\|\,\mathbf{x}\wedge\boldsymbol{\xi}\,\|_w}{\|\,\boldsymbol{\xi}\,\|_w}\Big)^{d_w/d} \prod_{v\neq w} \|\,\mathbf{x}\,\|_v^{d_v/d} \quad\text{and}\quad D_{\boldsymbol{\xi}}(\mathbf{x}) = \Big(\frac{|\mathbf{x}\cdot\boldsymbol{\xi}|_w}{\|\,\boldsymbol{\xi}\,\|_w}\Big)^{d_w/d} \prod_{v\neq w} \|\,\mathbf{x}\,\|_v^{d_v/d}.$$

In view of the product formula, these numbers depend only on the class of $\mathbf{x}$ in $\mathbb{P}^{n-1}(K)$. Clearly, they also depend only the class of $\boldsymbol{\xi}$ in $\mathbb{P}^{n-1}(K_w)$. So, in practice, we may always normalize $\boldsymbol{\xi}$ so that $\|\,\boldsymbol{\xi}\,\|_w = 1$.

Definition 2.1. We denote by $\widehat{\lambda}(\boldsymbol{\xi}, K, w)$ (resp. $\lambda(\boldsymbol{\xi}, K, w)$) the supremum of all real numbers λ for which the inequalities

$$H(\mathbf{x}) \le Q \quad\text{and}\quad D^*_{\boldsymbol{\xi}}(\mathbf{x}) \le Q^{-\lambda}$$

admit a non-zero solution $\mathbf{x} \in K^n$ for all sufficiently large (resp. for arbitrarily large) real numbers $Q \ge 1$. We also denote by $\widehat{\omega}(\boldsymbol{\xi}, K, w)$ (resp. $\omega(\boldsymbol{\xi}, K, w)$) the supremum of all real numbers ω for which the inequalities

$$H(\mathbf{x}) \le Q \quad\text{and}\quad D_{\boldsymbol{\xi}}(\mathbf{x}) \le Q^{-\omega}$$

admit a non-zero solution $\mathbf{x} \in K^n$ for all sufficiently large (resp. for arbitrarily large) real numbers $Q \ge 1$.

By construction, these numbers depend only on the class of $\boldsymbol{\xi}$ in $\mathbb{P}^{n-1}(K_w)$. Moreover, when $K = \mathbb{Q}$ and $w = \infty$, these are simply the standard exponents of approximation to a non-zero point $\boldsymbol{\xi} \in \mathbb{R}^n$ from the introduction. Indeed, each point of $\mathbb{P}^{n-1}(\mathbb{Q})$ is represented by a primitive integer point $\mathbf{x}$, that is a point of $\mathbb{Z}^n$ with relatively prime coordinates, and we have $H(\mathbf{x}) = \|\,\mathbf{x}\,\|$, $D^*_{\boldsymbol{\xi}}(\mathbf{x}) = \|\,\mathbf{x}\wedge\boldsymbol{\xi}\,\|$ and $D_{\boldsymbol{\xi}}(\mathbf{x}) = |\mathbf{x}\cdot\boldsymbol{\xi}|$ when $\|\,\boldsymbol{\xi}\,\| = 1$.

We can now state the main result of P. Bel in [1] to which we alluded in the introduction.

Theorem 2.2 (Bel, 2013). *Let $w \in M(K)$, and let S denote the set of elements of K_w^3 of the form $\boldsymbol{\xi} = (1, \xi, \xi^2)$ that have linearly independent coordinates over K. Then, the supremum of the numbers $\widehat{\lambda}(\boldsymbol{\xi}, K, w)$ with $\boldsymbol{\xi} \in S$ is $1/\gamma \simeq 0.618$ where $\gamma = (1+\sqrt{5})/2$ stands for the golden ratio.*

2.5. **Two dual families of minima.** Let w and $\boldsymbol{\xi} \in K_w^n$ be as in section 2.4. For each $j = 1, \dots, n$ and each $q \geq 0$, we define $L_{\boldsymbol{\xi},j}(q)$ (resp. $L^*_{\boldsymbol{\xi},j}(q)$) to be the smallest real number $t \geq 0$ for which the conditions

$$H(\mathbf{x}) \leq e^t \quad \text{and} \quad D_{\boldsymbol{\xi}}(\mathbf{x}) \leq e^{t-q} \quad \big(\text{resp. } D^*_{\boldsymbol{\xi}}(\mathbf{x}) \leq e^{t-q}\big) \tag{2.3}$$

admit at least j linearly independent solutions over K in K^n. This minimum exists since, for any number $B \geq 1$, there are only finitely many elements of $\mathbb{P}^{n-1}(K)$ of height at most B. We combine these functions into two maps

$$\mathbf{L}_{\boldsymbol{\xi}} = (L_{\boldsymbol{\xi},1}, \dots, L_{\boldsymbol{\xi},n}) \quad \text{and} \quad \mathbf{L}^*_{\boldsymbol{\xi}} = (L^*_{\boldsymbol{\xi},1}, \dots, L^*_{\boldsymbol{\xi},n})$$

from $[0, \infty)$ to $\mathbb{R}^n$. For $K = \mathbb{Q}$ and $K_w = \mathbb{R}$, the map $\mathbf{L}_{\boldsymbol{\xi}}$ is the same as in the introduction.

2.6. **The n-systems.** Let $q_0 \in [0, \infty)$. An n-system on $[q_0, \infty)$ is a continuous function $\mathbf{P} = (P_1, \dots, P_n)$ from $[q_0, \infty)$ to $\mathbb{R}^n$ with the following combinatorial properties.

(S1) For each $q \in [q_0, \infty)$, we have $0 \leq P_1(q) \leq \cdots \leq P_n(q)$ and $P_1(q) + \cdots + P_n(q) = q$.

(S2) There exists $s \in \{1, 2, \dots\} \cup \{\infty\}$ and a strictly increasing sequence $(q_i)_{0 \leq i < s}$ in $[q_0, \infty)$, which is unbounded if $s = \infty$, such that, over each subinterval $I_i = [q_{i-1}, q_i]$ with $1 \leq i < s$ including $I_s = [q_{s-1}, \infty)$ if $s < \infty$, the union of the graphs of $P_1, \dots, P_n$ decomposes as the union of $n - 1$ horizontal line segments and one line segment Γ_i of slope 1 (with possible crossings).

(S3) For each index i with $1 \leq i < s$, the line segment Γ_i ends strictly above the point where Γ_{i+1} starts (on the vertical line with abscissa q_i).

The sequence $(q_i)_{0 \leq i < s}$ is uniquely determined by $\mathbf{P}$. Its elements are called the *switch numbers* of $\mathbf{P}$. We say that an n-system is *rigid of mesh* c, for a given $c > 0$, if the n coordinates of $\mathbf{P}(q_i)$ are distinct positive multiples of c for each index i with $0 \leq i < s$. Then each q_i is also a positive multiple of c by the condition (S1). See [17, Figure 1] for a picture showing the combined graph of a rigid 5-system with $s = 3$.

For each n-system $\mathbf{P} = (P_1, \dots, P_n) \colon [q_0, \infty) \to \mathbb{R}^n$, we define its *dual* to be the map $\mathbf{P}^* \colon [q_0, \infty) \to \mathbb{R}^n$ given by

$$\mathbf{P}^*(q) = \big(q - P_n(q),\, q - P_{n-1}(q), \dots, q - P_1(q)\big) \quad \text{for each } q \geq q_0. \tag{2.4}$$

Note that $\mathbf{P}^*$ is not an n-system unless $n = 2$, in which case $\mathbf{P}^* = \mathbf{P}$.

2.7. **Main results.** With the notation of §2.5, we will show that the main result of parametric geometry of numbers from [17] extends naturally to the present more general setting. We state it below in dual form as well.

Theorem A. *Let $n \geq 2$ be an integer and let $w \in M(K)$. There are constants $c, c' > 0$ depending only on K, w and n with the following property. For each non-zero point $\boldsymbol{\xi} \in K_w^n$, there is an n-system $\mathbf{P}$ on $[0, \infty)$ such that*

$$\sup_{q \geq 0} \|\mathbf{L}_{\boldsymbol{\xi}}(q) - \mathbf{P}(q)\| \leq c \quad \textit{and} \quad \sup_{q \geq 0} \|\mathbf{L}^*_{\boldsymbol{\xi}}(q) - \mathbf{P}^*(q)\| \leq c \tag{2.5}$$

Conversely, for each n-system $\mathbf{P}$ on $[0,\infty)$, there is a non-zero point $\boldsymbol{\xi} \in K_w^n$ for which one of the two conditions in (2.5) *holds. Then the second condition holds with c replaced by c'.*

This means in particular that the two conditions in (2.5) are equivalent up to the value of the constant c. For $n = 1$, the statement of Theorem A is also true but not interesting because there is a unique 1-system $\mathbf{P}$ on $[0,\infty)$ and it satisfies $\mathbf{P}(q) = \mathbf{L}_{\boldsymbol{\xi}}(q) = q$ and $\mathbf{P}^*(q) = \mathbf{L}^*_{\boldsymbol{\xi}}(q) = 0$ for any $q \ge 0$ and any non-zero $\boldsymbol{\xi} \in K_w$. The next result deals with extension of scalars from $\mathbb{Q}$ to K.

Theorem B. *Let $n \ge 2$ be an integer, let $w \in M(K)$ be a place of K of relative degree $d_w = 1$ over a place ℓ of $\mathbb{Q}$, and let $\boldsymbol{\alpha} \in K^d$ be a basis of K over $\mathbb{Q}$. There is a constant $c'' > 0$ with the following property. For each non-zero $\boldsymbol{\xi} \in K_w^n$, the point $\Xi = \boldsymbol{\alpha} \otimes \boldsymbol{\xi} \in \mathbb{Q}_\ell^{nd}$ (defined in* (1.5)*) satisfies*

$$|L_{\Xi,d(i-1)+j}(dq) - L_{\boldsymbol{\xi},i}(q)| \le c'' \quad \text{and} \quad |L^*_{\Xi,d(i-1)+j}(dq) - L^*_{\boldsymbol{\xi},i}(q) - (d-1)q| \le c'' \tag{2.6}$$

for any choice of $q \ge 0$, $i = 1,\dots,n$ and $j = 1,\dots,d$.

Again the two sets of conditions in (2.6) in terms of the functions L and L^* are equivalent up to the value of c''. Our last main result provides very singular points on projective algebraic curves of degree $2d$ defined and irreducible over $\mathbb{Q}$.

Theorem C. *Suppose that K embeds in $\mathbb{Q}_\ell$ for a place ℓ of $\mathbb{Q}$. Identify K with its image and choose a basis $\boldsymbol{\alpha} \in \mathbb{Q}_\ell^d$ of K over $\mathbb{Q}$. Then we have*

$$\begin{aligned} &\sup\Big\{\widehat{\lambda}\big((\boldsymbol{\alpha},\xi\boldsymbol{\alpha},\xi^2\boldsymbol{\alpha}),\mathbb{Q},\ell\big)\,;\,\xi\in\mathbb{Q}_\ell \text{ and } [K(\xi):K]>2\Big\} = (d\gamma^2-1)^{-1},\\ &\sup\Big\{\widehat{\omega}\big((\boldsymbol{\alpha},\xi\boldsymbol{\alpha},\xi^2\boldsymbol{\alpha}),\mathbb{Q},\ell\big)\,;\,\xi\in\mathbb{Q}_\ell \text{ and } [K(\xi):K]>2\Big\} = d(\gamma^2+1)-1, \end{aligned} \tag{2.7}$$

where $\gamma = (1+\sqrt{5})/2$ stands for the Golden ratio.

Since $2 < \gamma^2 < 3$, this indeed provides very singular points $(\boldsymbol{\alpha},\xi\boldsymbol{\alpha},\xi^2\boldsymbol{\alpha}) \in \mathbb{Q}_\ell^{3d}$.

2.8. **Outline of the paper.** Most of the paper is devoted to the proof of Theorem A. This is done in two steps which we briefly sketch below.

We first show in section 7 how to attach an n-system to a non-zero point $\boldsymbol{\xi} \in K_w^n$. The general strategy is similar to that of Schmidt and Summerer in [22], instead that we need the adelic versions of Minkowski's convex body theorem and of Mahler's theory of compound bodies recalled in section 4. We also need a notion of distance $\lambda(\mathbf{x},\mathcal{C})$ between a non-zero point $\mathbf{x}$ of K^n and an adelic convex body $\mathcal{C}$, and a related notion of adelic minima for $\mathcal{C}$ defined in section 5. With those tools, we construct a family of adelic convex bodies $\mathcal{C}_{\boldsymbol{\xi}}(q)$ whose minima are closely related to the map $\mathbf{L}_{\boldsymbol{\xi}}(q)$, and we obtain information on this map by considering approximate compounds $\mathcal{C}^{(k)}_{\boldsymbol{\xi}}(q)$ of $\mathcal{C}_{\boldsymbol{\xi}}(q)$. The existence of an n-system that approximates $\mathbf{L}_{\boldsymbol{\xi}}(q)$ up to a bounded function then follows from a combinatorial result of [17] that is recalled in section 6.

The combinatorial result of section 6 is also used in order to attach a point $\boldsymbol{\xi}$ to an n-system. It shows that we simply have to do it for a rigid n-system $\mathbf{R}$ with a large mesh.

The construction of $\boldsymbol{\xi}$ is done in section 9. We work over the ring $\mathcal{O}_S$ of S-integers of K where S consists of w and all archimedean places of K. We construct recursively a sequence of ordered bases $\mathbf{x}^{(i)}$ of $\mathcal{O}_S^n$ over $\mathcal{O}_S$, one for each of the switch points q_i of $\mathbf{R}$. The basis $\mathbf{x}^{(i)}$ will realize, up to bounded factors, the successive minima of the adelic convex body $\mathcal{C}_{\boldsymbol{\xi}}(q)$ in the interval between q_i and q_{i+1} for the point $\boldsymbol{\xi}$ that we want (as illustrated for example in [17, Figure 2]). Each basis $\mathbf{x}^{(i)}$, except the first, is constructed from the preceding $\mathbf{x}^{(i-1)}$ by modifying only one point of it and by moving the new point up in the sequence, according to the behavior of the map $\mathbf{R}$ between q_{i-1} and q_i. This new point is obtained by multiplying the old one by an appropriate S-unit and by adding to this product a linear combination of some other points of $\mathbf{x}^{(i-1)}$ with coefficients in $\mathcal{O}_S$, in order to keep control on the geometry of $\mathbf{x}^{(i)}$ in K_v^n for each place v of S. For the places v distinct from w, this is done so that the image of $\mathbf{x}^{(i)}$ in K_v^n remains bounded and almost orthogonal in a sense that is defined in section 3. Meanwhile, at the place w, the norms of the basis elements of $\mathbf{x}^{(i)}$ in K_w^n are governed by the coordinates of $\mathbf{R}(q_i)$ and the subsequence of $\mathbf{x}^{(i)}$ common to $\mathbf{x}^{(i+1)}$ is almost orthogonal. Then the lines of K_w^n which are orthogonal to these subsequences for the dot product converge to a line whose generator $\boldsymbol{\xi}$ has the required property. The local estimates that are needed are developed in section 3, and the recursive procedure is presented in section 8, together with crucial estimates for local norms at the place w expressed in terms of heights only.

With the help of Theorem A, we show in section 10 that the spectrum of the four exponents $(\omega, \widehat{\omega}, \lambda, \widehat{\lambda})$ is independent of K and w and that it can be computed in terms of n-systems. We also extend the intermediate exponents of Laurent to the number field setting and derive the same conclusion for their spectrum.

Finally, Theorems B and C are proved in section 12 using a general construction in adelic geometry of numbers from section 11 that is reminiscent of work of Jeff Thunder in [24].

3. Local metric estimates

For the sake of generality, we fix here an arbitrary local field L, namely a complete field with respect to an absolute value $|\ |$ which either is archimedean or has a discrete valuation group $|L^*|$ in $\mathbb{R}^*$. For our applications this will be K_v for some place v of K. If L is archimedean, we identify it with $\mathbb{R}$ or $\mathbb{C}$ through an isometric field embedding in $\mathbb{C}$ (unique up to composition with complex conjugation). Otherwise, we denote by $\mathcal{O} = \{x \in K\,;\, |x| \le 1\}$ the valuation ring of L. In this section, we define notions of orthogonality and distance, and provide several estimates that will be needed in later sections (cf. [17, §4]).

3.1. **Norms and orthogonality.** Let k and n be integers with $1 \le k \le n$, and let U be a vector space over L of dimension n. If $L \subseteq \mathbb{C}$, we equip U with the euclidean norm associated to an inner product on U (real if $L = \mathbb{R}$ and complex if $L = \mathbb{C}$). Then there is a unique inner product on $\bigwedge^k U$ such that, for any orthonormal basis $(\mathbf{u}_1, \dots, \mathbf{u}_n)$ of U, the products $\mathbf{u}_{i_1} \wedge \cdots \wedge \mathbf{u}_{i_k}$ with $1 \le i_1 < \cdots < i_k \le n$ form an orthonormal basis of $\bigwedge^k U$, and we equip this space with the associated euclidean norm. If L is not archimedean, the ring $\mathcal{O}$

is a principal ideal domain and we equip U with the maximum norm with respect to some basis of U over L. Then, the unit ball $\mathcal{B}$ for that norm is the free rank n sub-$\mathcal{O}$-module of U generated by this basis and, for each $\mathbf{x} \in U$, we have

$$\| \mathbf{x} \| = \min\{|a|\,;\, a \in L \text{ and } \mathbf{x} \in a\mathcal{B}\}.$$

Moreover, the sub-$\mathcal{O}$-module $\bigwedge^k \mathcal{B}$ of $\bigwedge^k U$ generated by the products of k elements of $\mathcal{B}$ is free of rank $N = \binom{n}{k}$, and we equip $\bigwedge^k U$ with the corresponding norm.

If V is a subspace of U, we endow it with the induced norm. This norm is admissible because, if $L \not\subseteq \mathbb{C}$, it is associated to the sub-$\mathcal{O}$-module $\mathcal{B} \cap V$ of V which is free of rank $\dim_L(V)$. We say that subspaces $V_1, \dots, V_m$ of V are (topologically) *orthogonal* and, following the notation of [19, §§2.2], we write their sum as

$$V_1 \perp_{\mathrm{top}} \cdots \perp_{\mathrm{top}} V_m$$

if, for any choice of $(\mathbf{x}_1, \dots, \mathbf{x}_m) \in V_1 \times \cdots \times V_m$, we have

$$\| \mathbf{x}_1 + \cdots + \mathbf{x}_m \| = \begin{cases} (\| \mathbf{x}_1 \|^2 + \cdots + \| \mathbf{x}_m \|^2)^{1/2} & \text{if } L \subseteq \mathbb{C}, \\ \max\{\| \mathbf{x}_1 \|, \dots, \| \mathbf{x}_m \|\} & \text{otherwise.} \end{cases}$$

When $L \subseteq \mathbb{C}$, this is the usual notion and it amounts to asking that $V_1, \dots, V_m$ are pairwise orthogonal. However, when L is non-archimedean, the latter condition is necessary but not sufficient. We say that a point $\mathbf{x} \in U$ is *orthogonal* to a subspace V of U if $\langle \mathbf{x} \rangle_L$ and V are orthogonal. We say that an m-tuple of vectors $(\mathbf{x}_1, \dots, \mathbf{x}_m) \in U^m$ is *orthogonal* if the subspaces $\langle \mathbf{x}_1 \rangle_L, \dots, \langle \mathbf{x}_m \rangle_L$ that they span are orthogonal. We say that it is *orthonormal* if moreover they have norm 1. Again these are the usual notions when $L \subseteq \mathbb{C}$. When $L \not\subseteq \mathbb{C}$, an orthonormal basis of U is simply a basis of $\mathcal{B}$ as an $\mathcal{O}$-module. In general, an m-tuple of non-zero vectors $(\mathbf{x}_1, \dots, \mathbf{x}_m)$ of U is orthogonal (resp. orthonormal) if and only if it can be extended to an orthogonal (resp. orthonormal) basis $(\mathbf{x}_1, \dots, \mathbf{x}_n)$ of U. We will also need the following criterion.

Lemma 3.1. *With the above notation, let $\mathbf{x}_1, \dots, \mathbf{x}_m \in U \setminus \{0\}$. Then, we have*

$$\| \mathbf{x}_1 \wedge \cdots \wedge \mathbf{x}_m \| \le \| \mathbf{x}_1 \| \cdots \| \mathbf{x}_m \|$$

with equality if and only if $(\mathbf{x}_1, \dots, \mathbf{x}_m)$ is orthogonal.

On L^n we have the canonical bilinear form or dot product given by (2.1) for any pair of points $\mathbf{x} = (x_1, \dots, x_n)$ and $\mathbf{y} = (y_1, \dots, y_n)$ in L^n. If $L \subseteq \mathbb{C}$, this is connected with the inner product

$$(\mathbf{x}, \mathbf{y}) := \mathbf{x} \cdot \overline{\mathbf{y}} = x_1\overline{y}_1 + \cdots + x_n\overline{y}_n$$

where $\overline{\mathbf{y}} = (\overline{y}_1, \dots, \overline{y}_n)$ denotes the complex conjugate of $\mathbf{y}$, and we equip L^n with the corresponding euclidean norm. Otherwise, we equip L^n with the maximum norm, so that its unit ball is $\mathcal{O}^n$. When $L = K_v$, this agrees with the definitions of Section 2.2 both for the norm on L^n and the corresponding norm on $\bigwedge^k L^n$. We conclude with the following observation.

Lemma 3.2. *Let $(\mathbf{u}_1, \dots, \mathbf{u}_n)$ be an orthonormal basis of L^n. The dual basis $(\mathbf{u}_1^*, \dots, \mathbf{u}_n^*)$ of L^n with respect to the dot product is also orthonormal.*

Proof. If $L \subseteq \mathbb{C}$, then $\mathbf{u}_j^*$ is the complex conjugate $\overline{\mathbf{u}}_j$ of $\mathbf{u}_j$ for each $j = 1, \dots, n$, thus $(\mathbf{u}_i^*, \mathbf{u}_j^*) = \overline{(\mathbf{u}_i, \mathbf{u}_j)} = \delta_{i,j}$ for each $i, j \in \{1, \dots, n\}$, and we are done. If $L \not\subseteq \mathbb{C}$, then $(\mathbf{u}_1, \dots, \mathbf{u}_n)$ is a basis of $\mathcal{O}^n$ as an $\mathcal{O}$-module, thus $(\mathbf{u}_1^*, \dots, \mathbf{u}_n^*)$ is also a basis of $\mathcal{O}^n$ as needed. □

3.2. **Distances.** Again, let $1 \le k \le n$ be integers and let U be a vector space over L of dimension n equipped with an admissible norm, as above. By the choice of norm on L^n, a basis $(\mathbf{u}_1, \dots, \mathbf{u}_n)$ of U is orthonormal if and only if the linear map from L^n to U sending a point $(a_1, \dots, a_n) \in L^n$ to $a_1\mathbf{u}_1 + \cdots + a_n\mathbf{u}_n \in U$ is an isometry. We consider the following notions of distance.

Definition 3.3. The (projective) distance between non-zero points $\mathbf{x}$ and $\mathbf{y}$ of U, or between the lines $\langle \mathbf{x} \rangle_L$ and $\langle \mathbf{y} \rangle_L$ that they generate, is

$$\operatorname{dist}(\mathbf{x}, \mathbf{y}) := \operatorname{dist}(\langle \mathbf{x} \rangle_L, \langle \mathbf{y} \rangle_L) := \frac{\|\,\mathbf{x} \wedge \mathbf{y}\,\|}{\|\,\mathbf{x}\,\|\|\,\mathbf{y}\,\|} \in [0, 1].$$

If V_1, V_2 are subspaces of U of the same dimension k, then $\bigwedge^k V_1$, $\bigwedge^k V_2$ are one-dimensional subspaces of $\bigwedge^k U$, and we define

$$\operatorname{dist}(V_1, V_2) = \operatorname{dist}\left(\textstyle\bigwedge^k V_1, \bigwedge^k V_2\right).$$

As Schmidt notes in [20, §8], the distance between non-zero points of U satisfies the triangle inequality when $L \subseteq \mathbb{C}$. When L is non-archimedean, we state below a stronger inequality which we leave to the reader.

Lemma 3.4. *For any non-zero points $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3 \in U$, we have*

$$\operatorname{dist}(\mathbf{x}_1, \mathbf{x}_3) \le \begin{cases} \operatorname{dist}(\mathbf{x}_1, \mathbf{x}_2) + \operatorname{dist}(\mathbf{x}_2, \mathbf{x}_3) & \text{if } L \subseteq \mathbb{C}, \\ \max\{\operatorname{dist}(\mathbf{x}_1, \mathbf{x}_2),\, \operatorname{dist}(\mathbf{x}_2, \mathbf{x}_3)\} & \text{else.} \end{cases}$$

The same holds if $\mathbf{x}_j$ is replaced by a subspace V_j of U of dimension $k \ge 1$ for $j = 1, 2, 3$.

The second assertion of the lemma follows from the first when $k = 1$. The general case where $k > 1$ follows by considering the lines $\bigwedge^k V_j$ inside $\bigwedge^k U$.

For any subspace V of U, there is a subspace W of U such that $U = W \perp_{\mathrm{top}} V$. It suffices to choose an orthonormal basis $(\mathbf{u}_1, \dots, \mathbf{u}_k)$ for V (empty if $V = 0$), to complete it to an orthonormal basis $(\mathbf{u}_1, \dots, \mathbf{u}_n)$ of U, and to take $W = \langle \mathbf{u}_{k+1}, \dots, \mathbf{u}_n \rangle_L$. So, we may write any $\mathbf{x} \in U$ in the form $\mathbf{x} = \mathbf{w} + \mathbf{v}$ with $\mathbf{w} \in W$ orthogonal to V and $\mathbf{v} \in V$. If $L \subseteq \mathbb{C}$, this decomposition is unique. In general, it is not unique but the next result shows that $\|\,\mathbf{w}\,\|$ is independent of the decomposition (upon noting that $\mathbf{w} = 0$ when $\mathbf{x} = 0$).

Lemma 3.5. *Let V be a non-zero subspace of U and let $\mathbf{x} \in U \setminus \{0\}$. Write $\mathbf{x} = \mathbf{w} + \mathbf{v}$ with $\mathbf{w}$ orthogonal to V and $\mathbf{v} \in V$. Then we have*

$$\operatorname{dist}(\mathbf{x}, V) := \min\left\{ \operatorname{dist}(\mathbf{x}, \mathbf{y})\,;\, \mathbf{y} \in V \setminus \{0\} \right\} = \frac{\|\,\mathbf{w}\,\|}{\|\,\mathbf{x}\,\|}. \tag{3.1}$$

Proof. Let $\mathbf{y} \in V \setminus \{0\}$. Since $\mathbf{w}$ is orthogonal to V, it is orthogonal to $\mathbf{y}$. So, the products $\mathbf{w} \wedge \mathbf{y}$ and $\mathbf{v} \wedge \mathbf{y}$ are orthogonal in $\bigwedge^2 U$. Consequently, we find

$$\operatorname{dist}(\mathbf{x}, \mathbf{y}) = \frac{\| \mathbf{w} \wedge \mathbf{y} + \mathbf{v} \wedge \mathbf{y} \|}{\| \mathbf{x} \| \, \| \mathbf{y} \|} \geq \frac{\| \mathbf{w} \wedge \mathbf{y} \|}{\| \mathbf{x} \| \, \| \mathbf{y} \|} = \frac{\| \mathbf{w} \|}{\| \mathbf{x} \|},$$

with equality everywhere if $\mathbf{v} = 0$ or if $\mathbf{y} = \mathbf{v} \neq 0$. □

By Lemma 3.1, non-zero points $\mathbf{x}$, $\mathbf{y}$ of U are orthogonal if and only if $\operatorname{dist}(\mathbf{x}, \mathbf{y}) = 1$. Thus with the notation of Lemma 3.5, the point $\mathbf{x}$ is orthogonal to V if and only if $\operatorname{dist}(\mathbf{x}, V) = 1$. Moreover, we have $\mathbf{x} \in V$ if and only if $\operatorname{dist}(\mathbf{x}, V) = 0$. We also note the following alternative formula for $\operatorname{dist}(\mathbf{x}, V)$.

Lemma 3.6. *Let $\mathbf{x}$ and V be as in Lemma 3.5 and let $(\mathbf{y}_1, \dots, \mathbf{y}_k)$ be a basis of V. Then we have*

$$\operatorname{dist}(\mathbf{x}, V) = \frac{\| \mathbf{x} \wedge \mathbf{y}_1 \wedge \cdots \wedge \mathbf{y}_k \|}{\| \mathbf{x} \| \| \mathbf{y}_1 \wedge \cdots \wedge \mathbf{y}_k \|}. \tag{3.2}$$

Proof. Since the right hand side of (3.2) is independent of the choice of $(\mathbf{y}_1, \dots, \mathbf{y}_k)$, we may assume that this basis is orthogonal. Then, for the decomposition $\mathbf{x} = \mathbf{w} + \mathbf{v}$ of Lemma 3.5, the sequence $(\mathbf{w}, \mathbf{y}_1, \dots, \mathbf{y}_k)$ is also orthogonal. Thus, using Lemma 3.1, we find

$$\| \mathbf{x} \wedge \mathbf{y}_1 \wedge \cdots \wedge \mathbf{y}_k \| = \| \mathbf{w} \wedge \mathbf{y}_1 \wedge \cdots \wedge \mathbf{y}_k \| = \| \mathbf{w} \| \, \| \mathbf{y}_1 \wedge \cdots \wedge \mathbf{y}_k \|$$

and so the right hand side of (3.2) reduces to $\| \mathbf{w} \| / \| \mathbf{x} \| = \operatorname{dist}(\mathbf{x}, V)$. □

For the next crucial lemma, we apply the previous results with $U = L^n$.

Lemma 3.7. *Suppose that $n \geq m \geq 2$ for an integer m. Let V_1, V_2 be subspaces of L^n of dimension $m-1$ for which $W = V_1 \cap V_2$ has dimension at least $m-2$. Then we have*

$$\operatorname{dist}(V_1, V_2) = \max \big\{ \operatorname{dist}(\mathbf{x}, V_2) \, ; \, \mathbf{x} \in V_1 \setminus \{0\} \big\}. \tag{3.3}$$

Moreover, if $V_1 \neq V_2$, if $(\mathbf{w}_1, \dots, \mathbf{w}_{m-2})$ is a basis of W, and if $\mathbf{v}_i \in V_i \setminus W$ for $i = 1, 2$, then upon writing $\omega = \mathbf{w}_1 \wedge \cdots \wedge \mathbf{w}_{m-2}$ we have

$$\operatorname{dist}(V_1, V_2) = \frac{\| \omega \| \, \| \omega \wedge \mathbf{v}_1 \wedge \mathbf{v}_2 \|}{\| \omega \wedge \mathbf{v}_1 \| \, \| \omega \wedge \mathbf{v}_2 \|}. \tag{3.4}$$

Proof. We may assume that $V_1 \neq V_2$ since otherwise both sides of (3.3) are zero. We also note that the right hand side of (3.4) is independent of the choice of $\mathbf{w}_1, \dots, \mathbf{w}_{m-2}, \mathbf{v}_1, \mathbf{v}_2$. So, we choose for $(\mathbf{w}_1, \dots, \mathbf{w}_{m-2})$ an orthonormal basis of W and we complete it to an orthonormal basis $(\mathbf{w}_1, \dots, \mathbf{w}_{m-2}, \mathbf{v}_2, \mathbf{u})$ of $V_1 + V_2$ with $\mathbf{v}_2 \in V_2$. We also choose a unit vector $\mathbf{v}_1 \in V_1$ of the form $\mathbf{v}_1 = a\mathbf{v}_2 + b\mathbf{u}$ with $(a, b) \in L^2$. Then $(\mathbf{w}_1, \dots, \mathbf{w}_{m-2}, \mathbf{v}_i)$ is an orthonormal basis of V_i for $i = 1, 2$ and we have $\| (a, b) \| = 1$. Moreover, the pair $(\omega \wedge \mathbf{v}_2, \omega \wedge \mathbf{u})$ is orthonormal in $\bigwedge^{m-1} L^n$. Since $\omega \wedge \mathbf{v}_1 = a\omega \wedge \mathbf{v}_2 + b\omega \wedge \mathbf{u}$, we deduce that

$$\operatorname{dist}(V_1, V_2) = \operatorname{dist}(\omega \wedge \mathbf{v}_1, \omega \wedge \mathbf{v}_2) = |b|.$$

This proves (3.4) since $\|\,\omega\wedge\mathbf{v}_1\wedge\mathbf{v}_2\,\| = \|\,b\omega\wedge\mathbf{u}\wedge\mathbf{v}_2\,\| = |b|$. Finally, let $\mathbf{x}\in V_1\setminus\{0\}$, and write $\mathbf{x}=\mathbf{w}+t\mathbf{v}_1$ with $\mathbf{w}\in W$ and $t\in L$. Then, $\mathbf{x}=tb\mathbf{u}+\mathbf{v}$ where $\mathbf{v}=\mathbf{w}+ta\mathbf{v}_2\in V_2$ and $\mathbf{u}$ is orthogonal to V_2. So, Lemma 3.5 gives

$$\operatorname{dist}(\mathbf{x},V_2)=\frac{|tb|}{\|\,\mathbf{x}\,\|}\le\frac{|tb|}{|t|}=|b|$$

with equality if $\mathbf{x}=\mathbf{v}_1$. This proves (3.3). □

Corollary 3.8. *Let V_1, V_2 be as in Lemma 3.7 and let $\mathbf{x}\in L^n\setminus\{0\}$. Then we have*

$$\operatorname{dist}(\mathbf{x},V_2)\le\begin{cases}\operatorname{dist}(\mathbf{x},V_1)+\operatorname{dist}(V_1,V_2) & \text{if } L\subseteq\mathbb{C},\\ \max\{\operatorname{dist}(\mathbf{x},V_1),\operatorname{dist}(V_1,V_2)\} & \text{else.}\end{cases}$$

Proof. Choose $\mathbf{y}_1\in V_1\setminus\{0\}$ such that $\operatorname{dist}(\mathbf{x},\mathbf{y}_1)=\operatorname{dist}(\mathbf{x},V_1)$ and $\mathbf{y}_2\in V_2\setminus\{0\}$ such that $\operatorname{dist}(\mathbf{y}_1,\mathbf{y}_2)=\operatorname{dist}(\mathbf{y}_1,V_2)$. By Lemma 3.7, we have $\operatorname{dist}(\mathbf{y}_1,\mathbf{y}_2)\le\operatorname{dist}(V_1,V_2)$. As $\operatorname{dist}(\mathbf{x},V_2)\le\operatorname{dist}(\mathbf{x},\mathbf{y}_2)$, the conclusion follows from the triangle inequality of Lemma 3.4 applied to $\mathbf{x}$, $\mathbf{y}_1$ and $\mathbf{y}_2$. □

3.3. **Duality.** For each $k=0,\dots,n$, the dot product on $L^n=\bigwedge^1 L^n$ induces a non-degenerate bilinear map from $\bigwedge^k L^n\times\bigwedge^k L^n$ to L also denoted by a dot and given on pure products by

$$(\mathbf{x}_1\wedge\cdots\wedge\mathbf{x}_k)\cdot(\mathbf{y}_1\wedge\cdots\wedge\mathbf{y}_k)=\det(\mathbf{x}_i\cdot\mathbf{y}_j)$$

with the convention that, for $k=0$, the empty wedge product is $1\in L=\bigwedge^0 L^n$ and the empty determinant is 1 as well.

Let $(\mathbf{e}_1,\dots,\mathbf{e}_n)$ denote the canonical basis of L^n and let $\mathbf{E}=\mathbf{e}_1\wedge\cdots\wedge\mathbf{e}_n$. For k as above, there is a unique isomorphism $\varphi_k\colon\bigwedge^k L^n\to\bigwedge^{n-k}L^n$ such that

$$(\mathbf{X}\wedge\mathbf{Y})\cdot\mathbf{E}=\varphi_k(\mathbf{X})\cdot\mathbf{Y}$$

for any $\mathbf{X}\in\bigwedge^k L^n$ and $\mathbf{Y}\in\bigwedge^{n-k}L^n$. If $(\mathbf{u}_1,\dots,\mathbf{u}_n)$ is any basis of L^n with $\mathbf{u}_1\wedge\cdots\wedge\mathbf{u}_n=\mathbf{E}$, and if $(\mathbf{u}_1^*,\dots,\mathbf{u}_n^*)$ denotes the dual basis of L^n for the dot product, a short computation shows that, for any k-tuple of integers $\mathbf{i}=(i_1,\dots,i_k)$ with $1\le i_1<\cdots<i_k\le n$, we have

$$\varphi_k(\mathbf{u}_{i_1}\wedge\cdots\wedge\mathbf{u}_{i_k})=\epsilon(\mathbf{i},\mathbf{j})\mathbf{u}_{j_1}^*\wedge\cdots\wedge\mathbf{u}_{j_{n-k}}^*\tag{3.5}$$

where $\mathbf{j}=(j_1,\dots,j_{n-k})$ denotes the complementary increasing sequence of integers for which $(\mathbf{i},\mathbf{j})$ is a permutation of $(1,\dots,n)$, and $\epsilon(\mathbf{i},\mathbf{j})\in\{-1,1\}$ is the signature of this permutation. In particular, if we choose $(\mathbf{u}_1,\dots,\mathbf{u}_n)$ to be the canonical basis of L^n, which is its own dual, this formula shows that φ_k is an isometry. Furthermore, if V is a subspace of L^n of dimension k, we may choose $(\mathbf{u}_1,\dots,\mathbf{u}_n)$ so that $(\mathbf{u}_1,\dots,\mathbf{u}_k)$ is a basis of V. Then $(\mathbf{u}_{k+1}^*,\dots,\mathbf{u}_n^*)$ is a basis of the subspace

$$V^\perp=\{\mathbf{y}\in L^n\,;\,\mathbf{x}\cdot\mathbf{y}=0\text{ for each }\mathbf{x}\in V\}$$

and the same formula implies that

$$\varphi_k(\textstyle\bigwedge^k V)=\bigwedge^{n-k}V^\perp.$$

As φ_k is an isometry, it preserves the distance and so we conclude as follows.

Lemma 3.9. *For any pair of subspaces V_1, V_2 of L^n of the same dimension k with $0<k<n$, we have*

$$\operatorname{dist}(V_1,V_2)=\operatorname{dist}(V_1^\perp,V_2^\perp).$$

In particular, if V_1, V_2 have dimension $n-1>0$ and if $V_i^\perp=\langle \mathbf{u}_i\rangle_L$ for $i=1,2$, then $\operatorname{dist}(V_1,V_2)=\operatorname{dist}(\mathbf{u}_1,\mathbf{u}_2)$. When $L=\mathbb{R}$, this observation also follows from [17, Lemma 4.4].

3.4. **Almost orthogonal sequences.** We set

$$\delta=\begin{cases}1 & \text{if } L=\mathbb{C},\\ 0 & \text{otherwise.}\end{cases}\tag{3.6}$$

and say that a non-empty sequence $(\mathbf{x}_1,\dots,\mathbf{x}_m)$ in L^n is *almost orthogonal* if it is linearly independent over L and satisfies

$$\operatorname{dist}(\mathbf{x}_j,\langle \mathbf{x}_1,\dots,\mathbf{x}_{j-1}\rangle_L)\ge 1-\delta/2^{j-1}\quad (2\le j\le m).$$

Thus almost orthogonal means orthogonal when L is non-archimedean. As in [17, §4], we note that any non-empty subsequence of an almost orthogonal sequence is almost orthogonal. Since $\prod_{j\ge 2}(1-\delta/2^{j-1})\ge e^{-2\delta}$, we find the following estimate (cf. [17, Lemma 4.6]).

Lemma 3.10. *For any almost orthogonal sequence $(\mathbf{x}_1,\dots,\mathbf{x}_m)$ in L^n we have*

$$e^{-2\delta}\|\mathbf{x}_1\|\cdots\|\mathbf{x}_m\|\le\|\mathbf{x}_1\wedge\cdots\wedge\mathbf{x}_m\|\le\|\mathbf{x}_1\|\cdots\|\mathbf{x}_m\|.$$

The next crucial result is analogous to [17, Lemma 4.7].

Lemma 3.11. *Let k, ℓ, m be integers with $1\le k<\ell\le m\le n$ and let $\mathbf{y}_1,\dots,\mathbf{y}_m$ be linearly independent points of L^n. Suppose that the sequences $(\mathbf{y}_1,\dots,\widehat{\mathbf{y}_\ell},\dots,\mathbf{y}_m)$ and $(\mathbf{y}_1,\dots,\widehat{\mathbf{y}_k},\dots,\mathbf{y}_m)$ are both almost orthogonal. Then, the subspaces*

$$V_1=\langle \mathbf{y}_1,\dots,\widehat{\mathbf{y}_\ell},\dots,\mathbf{y}_m\rangle_L\quad\text{and}\quad V_2=\langle \mathbf{y}_1,\dots,\widehat{\mathbf{y}_k},\dots,\mathbf{y}_m\rangle_L$$

that they span in L^n satisfy

$$\operatorname{dist}(V_1,V_2)\le e^{4\delta}\frac{\|\mathbf{y}_1\wedge\cdots\wedge\mathbf{y}_m\|}{\|\mathbf{y}_1\|\cdots\|\mathbf{y}_m\|}.$$

Proof. Upon setting $\omega=\mathbf{y}_1\wedge\cdots\wedge\widehat{\mathbf{y}_k}\wedge\cdots\wedge\widehat{\mathbf{y}_\ell}\wedge\cdots\wedge\mathbf{y}_m$, Lemma 3.7 gives

$$\operatorname{dist}(V_1,V_2)=\frac{\|\omega\|\,\|\mathbf{y}_1\wedge\cdots\wedge\mathbf{y}_m\|}{\|\omega\wedge\mathbf{y}_\ell\|\,\|\omega\wedge\mathbf{y}_k\|}.$$

The conclusion follows because, by Lemma 3.10,

$$\begin{aligned}\|\omega\wedge\mathbf{y}_\ell\|\,\|\omega\wedge\mathbf{y}_k\|&\ge e^{-4\delta}\big(\|\mathbf{y}_1\|\cdots\widehat{\|\mathbf{y}_k\|}\cdots\|\mathbf{y}_m\|\big)\big(\|\mathbf{y}_1\|\cdots\widehat{\|\mathbf{y}_\ell\|}\cdots\|\mathbf{y}_m\|\big)\\&\ge e^{-4\delta}\|\omega\|\,\big(\|\mathbf{y}_1\|\cdots\|\mathbf{y}_m\|\big).\end{aligned}$$

□

We conclude with a simple estimate.

Lemma 3.12. *For any unit vectors $\mathbf{u},\mathbf{u}'\in L^n$ and any $\mathbf{x}\in L^n$, we have*

$$|\mathbf{x}\cdot\mathbf{u}|\le 2^\delta\max\{|\mathbf{x}\cdot\mathbf{u}'|,\ \|\mathbf{x}\|\operatorname{dist}(\mathbf{u},\mathbf{u}')\}.\tag{3.7}$$

Proof. We have $\| (\mathbf{x}\cdot\mathbf{u})\mathbf{u}' - (\mathbf{x}\cdot\mathbf{u}')\mathbf{u}\| \le \|\mathbf{x}\|\,\|\mathbf{u}\wedge\mathbf{u}'\|$ for any $\mathbf{x},\mathbf{u},\mathbf{u}'\in L^n$. □

When $\mathbf{x}\cdot\mathbf{u}'=0$, this becomes simply $|\mathbf{x}\cdot\mathbf{u}| \le 2^\delta \|\mathbf{x}\|\operatorname{dist}(\mathbf{u},\mathbf{u}')$.

4. Adelic geometry of numbers

For each place v of K, we form the compact set

$$\mathcal{O}_v = \{x\in K_v\,;\, |x|_v\le 1\}.$$

When $v\nmid\infty$, this is the ring of integers of K_v, and we follow MacFeat [13] in normalizing the Haar measure μ_v on K_v so that $\mu_v(\mathcal{O}_v)=1$. When $v\mid\infty$, we set μ_v to be the Lebesgue measure on K_v, with $\mu_v(\mathcal{O}_v)=2$ if $K_v=\mathbb{R}$, and $\mu_v(\mathcal{O}_v)=\pi$ if $K_v=\mathbb{C}$. We also denote by μ_v the product measure on K_v^n.

The ring of adèles of K is the subring $K_{\mathbb{A}}$ of $\prod_{v\in M(K)} K_v$ which consists of the sequences (a_v) with $a_v\in\mathcal{O}_v$ for all but finitely v. It is endowed with the unique topology which extends the product topology on the set $\mathcal{O}_{\mathbb{A}} := \prod_{v\mid\infty} K_v \times \prod_{v\nmid\infty}\mathcal{O}_v$, and makes $K_{\mathbb{A}}$ into a locally compact ring with $\mathcal{O}_{\mathbb{A}}$ as an open subring. Then K embeds in $K_{\mathbb{A}}$ as a discrete subring via the diagonal map. We denote by μ the Haar measure on $K_{\mathbb{A}}$ whose restriction to $\mathcal{O}_{\mathbb{A}}$ is the product of the μ_v, and we use the same notation for the product measure on $K_{\mathbb{A}}^n$.

When $v\mid\infty$, a (Minkowski) convex body of K_v^n is any compact convex neighborhood $\mathcal{C}_v$ of 0 such that $a\,\mathcal{C}_v\subseteq\mathcal{C}_v$ for each $a\in\mathcal{O}_v$. When $v\nmid\infty$, this is any finitely generated (thus free and compact) $\mathcal{O}_v$-submodule $\mathcal{C}_v$ of K_v^n of rank n. Finally, a convex body of $K_{\mathbb{A}}^n$ is any product $\mathcal{C}=\prod_v\mathcal{C}_v$ where $\mathcal{C}_v$ is a convex body of K_v^n for each v, and $\mathcal{C}_v=\mathcal{O}_v^n$ for all but finitely v. Then the induced topology on $\mathcal{C}$ coincides with the usual product topology, and its volume $\mu(\mathcal{C})=\prod_v\mu_v(\mathcal{C}_v)$ is finite and positive.

For each $j=1,\dots,n$, we define the j-th minimum $\lambda_j(\mathcal{C})$ of a convex body $\mathcal{C}=\prod_v\mathcal{C}_v$ of $K_{\mathbb{A}}^n$ to be the smallest $\lambda>0$ for which the dilated convex body

$$\lambda\,\mathcal{C} = \prod_{v\mid\infty}(\lambda\,\mathcal{C}_v)\prod_{v\nmid\infty}\mathcal{C}_v$$

contains at least j linearly independent elements of K^n over K. With this notation, the adelic version of Minkowski's theorem reads as follows [13, 2].

Theorem 4.1 (McFeat, 1971; Bombieri and Vaaler, 1983)**.** *For any convex body $\mathcal{C}$ of $K_{\mathbb{A}}^n$, we have*

$$(\lambda_1(\mathcal{C})\cdots\lambda_n(\mathcal{C}))^d\,\mu(\mathcal{C})\asymp 1,$$

with implicit constants that depend only on K and n.

We refer the reader to [13, Theorems 5 and 6], [2, Theorems 3 and 6] and [24, Corollary of Theorem 1] for explicit lower bounds and upper bounds. In particular this result implies that, if the volume $\mu(\mathcal{C})$ of $\mathcal{C}$ is large enough, then $\lambda_1(\mathcal{C})\le 1$ and so $\mathcal{C}$ contains a non-zero point of K^n.

More generally, fix an integer k with $1 \le k \le n$ and set $N = \binom{n}{k}$. The K-linear isomorphism from $\bigwedge^k K^n$ to K^N that sends a point to its Plücker coordinates extends to a K_v-linear topological isomorphism from $\bigwedge^k K_v^n$ to K_v^N and to a to a $K_{\mathbb{A}}$-linear topological isomorphism from $\bigwedge^k K_{\mathbb{A}}^n$ to $K_{\mathbb{A}}^N$. Identifying these pairs of spaces, we obtain a measure μ_v on $\bigwedge^k K_v^n$ for each $v \in M(K)$ and a measure μ on $\bigwedge^k K_{\mathbb{A}}^n$. This also provides the notion of a (Minkowski) convex body $\mathcal{K}_v$ of $\bigwedge^k K_v^n$ for each $v \in M(K)$ and of a convex body $\mathcal{K} = \prod_v \mathcal{K}_v$ of $\bigwedge^k K_{\mathbb{A}}^n$, as well as the notion of the j-th minimum $\lambda_j(\mathcal{K})$ of $\mathcal{K}$ with respect to $\bigwedge^k K^n$, for each $j = 1, \dots, N$.

Let $\mathcal{C} = \prod_v \mathcal{C}_v$ be a convex body of $K_{\mathbb{A}}^n$. Its k-th compound is the convex body $\bigwedge^k \mathcal{C} = \prod_v (\bigwedge^k \mathcal{C}_v)$ of $\bigwedge^k K_{\mathbb{A}}^n$ whose component $\bigwedge^k \mathcal{C}_v$ at a place v is the smallest Minkowski convex body of $\bigwedge^k K_v^n$ containing all products $\mathbf{x}_1 \wedge \cdots \wedge \mathbf{x}_k$ of k elements $\mathbf{x}_1, \dots, \mathbf{x}_k$ of $\mathcal{C}_v$. In particular, we have $\bigwedge^1 \mathcal{C} = \mathcal{C}$. In this context, E. B. Burger has extended Mahler's theory of compound bodies in [5]. Leaving out the explicit values of the constants from [5, Theorem 1.2], he showed that the minima of these convex bodies are related as follows.

Theorem 4.2 (Burger, 1993)**.** *With the above notation, order the N products $\lambda_{i_1}(\mathcal{C}) \cdots \lambda_{i_k}(\mathcal{C})$ with $1 \le i_1 < \cdots < i_k \le n$ into a monotonically increasing sequence $\Lambda_1 \le \cdots \le \Lambda_N$. Then, for each $j = 1, \dots, N$, we have*

$$\lambda_j \left(\bigwedge\nolimits^k \mathcal{C} \right) \asymp \Lambda_j$$

with implicit constants depending only on K and n.

We note that $\Lambda_1 = \lambda_1(\mathcal{C}) \cdots \lambda_k(\mathcal{C})$, and that $\Lambda_2 = \lambda_1(\mathcal{C}) \cdots \lambda_{k-1}(\mathcal{C}) \lambda_{k+1}(\mathcal{C})$ if $k < n$. Moreover, if $\mathbf{x}_1, \dots, \mathbf{x}_n$ are linearly independent elements of K^n which realize the successive minima of $\mathcal{C}$ in the sense that $\mathbf{x}_i \in \lambda_i(\mathcal{C})\mathcal{C}$ for $i = 1, \dots, n$, then $\mathbf{X} = \mathbf{x}_1 \wedge \cdots \wedge \mathbf{x}_k$ belongs to $\Lambda_1 \bigwedge^k \mathcal{C}$. Thus, by the above theorem, the first minimum of $\bigwedge^k \mathcal{C}$ is realized up to a bounded factor by the pure product $\mathbf{X}$.

In practice, the compounds of a given convex body are difficult to compute exactly. So, we instead use approximations of them, like in the standard theory (see [21, Chapter IV, §7]).

5. Dilations

The group of idèles of K is the group $K_{\mathbb{A}}^*$ of invertible elements of $K_{\mathbb{A}}$. It contains the multiplicative group K^* of K as a subgroup. We define the module $|\boldsymbol{a}|_{\mathbb{A}}$ of an idèle $\boldsymbol{a} = (a_v) \in K_{\mathbb{A}}^*$ by

$$|\boldsymbol{a}|_{\mathbb{A}} = \prod_{v \in M(K)} |a_v|_v^{d_v/d} \in \mathbb{R}_{>0},$$

and recall that $|\alpha|_{\mathbb{A}} = 1$ for any $\alpha \in K^*$. Then for each convex body $\mathcal{C} = \prod_v \mathcal{C}_v$ of $K_{\mathbb{A}}^n$, the product

$$\boldsymbol{a}\,\mathcal{C} = \prod_{v \in M(K)} (a_v\,\mathcal{C}_v)$$

is a convex body of volume $\mu(\boldsymbol{a}\,\mathcal{C}) = |\boldsymbol{a}|_{\mathbb{A}}^{dn}\mu(\mathcal{C})$. This construction extends the definition of $\lambda\mathcal{C}$ with $\lambda \in \mathbb{R}_{>0}$ by identifying any such λ with the idèle having component λ at each Archimedean place $v \mid \infty$ and component 1 at all other places.

Definition 5.1. For $\mathcal{C}$ as above and for each non-zero $\mathbf{x} \in K^n$, we set

$$\lambda(\mathbf{x}, \mathcal{C}) = \min\{|\boldsymbol{a}|_{\mathbb{A}}\,;\, \boldsymbol{a} \in K_{\mathbb{A}}^* \text{ and } \mathbf{x} \in \boldsymbol{a}\mathcal{C}\}. \tag{5.1}$$

For each $j = 1, \dots, n$, we also define $\lambda_j^{\mathbb{A}}(\mathcal{C})$ to be the smallest $\lambda > 0$ for which there are at least j linearly independent elements $\mathbf{x}$ of K^n with $\lambda(\mathbf{x}, \mathcal{C}) \le \lambda$.

The minimum exists in (5.1) because for each $v \in M(K)$ there is a non-zero $a_v \in K_v$ with $|a_v|_v$ minimal such that $\mathbf{x} \in a_v\mathcal{C}_v$, and we may choose $a_v = 1$ for all but finitely many v. Moreover, by the product formula, the value $\lambda(\mathbf{x}, \mathcal{C})$, which we view as a sort of distance from $\mathbf{x}$ to $\mathcal{C}$, depends only on the class of $\mathbf{x}$ in $\mathbb{P}^{n-1}(K)$. In particular, it is independent of $\mathbf{x}$ if $n = 1$. We also note that, for any given $t > 0$, the non-zero points $\mathbf{x}$ of K^n with $\lambda(\mathbf{x}, \mathcal{C}) \le t$ have height at most ct for a constant $c > 0$ depending only on $\mathcal{C}$. So these points $\mathbf{x}$ belong to finitely many classes in $\mathbb{P}^{n-1}(K)$ and for them $\lambda(\mathbf{x}, \mathcal{C})$ takes finitely many values in $[0, t]$. Hence, there is a basis $(\mathbf{x}_1, \dots, \mathbf{x}_n)$ of K^n over K such that $\lambda_j^{\mathbb{A}}(\mathcal{C}) = \lambda(\mathbf{x}_j, \mathcal{C})$ for each $j = 1, \dots, n$. In particular, it is sensible to define each $\lambda_j^{\mathbb{A}}(\mathcal{C})$ as a minimum.

To compare these minima to those of MacFeat and Bombieri–Vaaler, we need the following special case of the strong approximation theorem from [10, Theorem 3].

Lemma 5.2. *There exists a constant $c_1 = c_1(K) > 0$ with the following property. For each $\boldsymbol{a} = (a_v) \in K_{\mathbb{A}}^*$ with $|\boldsymbol{a}|_{\mathbb{A}} \ge c_1$, there exists $\alpha \in K^*$ such that $|\alpha|_v \le |a_v|_v$ for each $v \in M(K)$.*

Note that this also follows the adelic version of Minkowski's theorem, because, for given $\boldsymbol{a} = (a_v) \in K_{\mathbb{A}}^*$, the set of points $(x_v) \in K_{\mathbb{A}}$ with $|x_v|_v \le |a_v|_v$ for all v is the convex body $\boldsymbol{a}\,\mathcal{B}$ of $K_{\mathbb{A}}$ of volume $|\boldsymbol{a}|_{\mathbb{A}}^d\mu(\mathcal{B})$, where $\mathcal{B} = \prod_v \mathcal{O}_v$. So, if $|\boldsymbol{a}|_{\mathbb{A}}$ is large enough, Theorem 4.1 gives $\lambda_1(\boldsymbol{a}\,\mathcal{B}) \le 1$, and thus $\boldsymbol{a}\,\mathcal{B}$ contains some non-zero element of K.

Proposition 5.3. *Let $\mathcal{C}$ be a convex body of $K_{\mathbb{A}}^n$ and let $j \in \{1, \dots, n\}$. Then, we have*

$$c_1^{-1}\lambda_j(\mathcal{C}) \le \lambda_j^{\mathbb{A}}(\mathcal{C}) \le \lambda_j(\mathcal{C}) \tag{5.2}$$

where c_1 comes from Lemma 5.2. Moreover, for each idèle $\boldsymbol{a} \in K_{\mathbb{A}}^*$, we also have

$$\lambda_j^{\mathbb{A}}(\boldsymbol{a}\,\mathcal{C}) = |\boldsymbol{a}|_{\mathbb{A}}^{-1}\lambda_j^{\mathbb{A}}(\mathcal{C}). \tag{5.3}$$

Proof. Set $\lambda = \lambda_j^{\mathbb{A}}(\mathcal{C})$ and choose a set F of j linearly independent points $\mathbf{x}$ of K^n with $\lambda(\mathbf{x}, \mathcal{C}) \le \lambda$. Given $\mathbf{x} \in F$, there exists $\boldsymbol{a} \in K_{\mathbb{A}}^*$ with $|\boldsymbol{a}|_{\mathbb{A}} \le \lambda$ such that $\mathbf{x} \in \boldsymbol{a}\mathcal{C}$. As the idèle $\boldsymbol{a}' = (a'_v) := c_1\lambda\boldsymbol{a}^{-1}$ satisfies $|\boldsymbol{a}'|_{\mathbb{A}} \ge c_1$, Lemma 5.2 provides $\alpha \in K^*$ such that $|\alpha|_v \le |a'_v|_v$ for each $v \in M(K)$, and then the point $\alpha\mathbf{x}$ of K^n belongs to $\alpha\boldsymbol{a}\mathcal{C} \subseteq \boldsymbol{a}'\boldsymbol{a}\mathcal{C} = c_1\lambda\mathcal{C}$. Doing this for each $\mathbf{x} \in F$, we obtain j linearly independent points of K^n in $c_1\lambda\mathcal{C}$. This means that $\lambda_j(\mathcal{C}) \le c_1\lambda$, which amounts to the first inequality in (5.2).

To prove the second inequality in (5.2), set $\lambda = \lambda_j(\mathcal{C})$. Then $\lambda\mathcal{C}$ contains at least j linearly independent elements of K^n. As $|\lambda|_{\mathbb{A}} = \lambda$, this implies that $\lambda_j^{\mathbb{A}}(\mathcal{C}) \le \lambda$ and we are done.

Finally, the inequality (5.3) follows from the definitions and the multiplicativity of the module on $K_{\mathbb{A}}^*$. $\square$

In view of our identifications (see section 4), the above results and definitions apply with $\mathcal{C}$ replaced by any convex body $\mathcal{K} = \prod_v \mathcal{K}_v$ of $\bigwedge^k K_{\mathbb{A}}^n$ for any integer $1 \le k \le n$, provided that n is replaced by $N = \binom{n}{k}$ and that K^n is replaced by $\bigwedge^k K^n$.

6. A combinatorial result

In preparation for the proof of Theorem A in the next sections, we will need the following result from [17]. We refer the reader to Section 2.6 for the definition of an n-system.

Proposition 6.1. *Let $c \ge 0$. Suppose that, for each $k = 1, \dots, n$, there are continuous functions $L_k\colon [0,\infty) \to \mathbb{R}$ and $M_k\colon [0,\infty) \to \mathbb{R}$ which are piecewise linear with slopes 0 and 1, and which satisfy the following properties:*

(1) $0 \le L_1(q) \le \cdots \le L_n(q) \le q$ *for each* $q \ge 0$*;*

(2) $|M_k(q) - L_1(q) - \cdots - L_k(q)| \le c$ *for each* $k = 1, \dots, n$ *and each* $q \ge 0$*;*

(3) $M_n(q) = q$ *for each* $q \ge 0$*;*

(4) *if, for some integer k with $1 \le k < n$ and some $q > 0$, the function M_k changes slope from 1 to 0 at q, then $|L_{k+1}(q) - L_k(q)| \le 2c$.*

Choose $c' > 24n^3c$ and set

$$t_i = (1 + 2 + \cdots + i)c' \quad \text{for } i = 0, 1, \dots, n.$$

Then there exists an n-system $\mathbf{R} = (R_1, \dots, R_n)$ on $[0,\infty)$, whose restriction to $[t_n,\infty)$ is rigid of mesh c', such that

(5) $\max_{1\le k\le n} |L_k(q) - R_k(q)| \le 4n^2c'$ *for each* $q \ge 0$*;*

(6) $\mathbf{R}(t_i) = \big(0, \dots, 0, c', 2c', \dots, ic'\big)$ *for each* $i = 0, 1, \dots, n$*.*

Proof. Define $M_0 = 0$ and $P_k = M_k - M_{k-1}$ for $k = 1, \dots, n$. Put also $\gamma = 6c$. By adapting the proof of [17, Theorem 2.9], we find that $\mathbf{P} = (P_1, \dots, P_n)\colon [0,\infty) \to \mathbb{R}^n$ is an (n,γ)-system in the sense of [17, Definition 2.8], with $|L_k(q) - P_k(q)| \le \gamma$ for each $q \ge 0$ and $k = 1, \dots, n$. Then, arguing as in the proof of [17, Theorem 8.2], we obtain a rigid n-system $\mathbf{R} = (R_1, \dots, R_n)\colon [t_n,\infty) \to \mathbb{R}^n$ of mesh c' which satisfies the condition (5) for each $q \ge t_n$. In particular, $\mathbf{R}(t_n)$ is a strictly increasing sequence of positive integer multiples of c' with sum t_n, and so $\mathbf{R}(t_n) = (c', 2c', \dots, nc')$. From this it follows that $\mathbf{R}$ extends uniquely to an n-system on $[0,\infty)$ satisfying the condition (6) (see the proof of [17, Theorem 8.1]). For each $k = 1, \dots, n$ and each $q \ge 0$, we have $0 \le L_k(q), R_k(q) \le q$, thus $|L_k - R_k|$ is bounded above by $\max\{t_n, 4n^2c'\} = 4n^2c'$ on $[0,\infty)$. $\square$

Note that, for $c = 0$, the hypotheses of Proposition 6.1 amount to asking that the map $\mathbf{L} := (L_1, \dots, L_n)$ itself is an n-system on $[0,\infty)$ (and that $M_k = L_1 + \cdots + L_k$ for each $k = 1, \dots, n$). In fact, this is how n-systems are defined in [17, §2.5] (where they are called $(n,0)$-systems). From this, we infer the following result of approximation.

Corollary 6.2. *Let $c' > 0$, let $q_0 = (n^2-2n+1)c'/2$, and let $\mathbf{L} = (L_1, \dots, L_n)$ be an n-system on $[0,\infty)$. Then, there exists an n-system $\mathbf{R} = (R_1, \dots, R_n)$ on $[0,\infty)$ whose restriction to $[q_0,\infty)$ is rigid of mesh $c'/2$, which satisfies $\max_{1\le k\le n} |L_k(q)-R_k(q)| \le 4n^2c'$ for each $q \ge 0$, and for which R_1 has slope 1 on $[q_0, q_0 + c'/2]$.*

Proof. The conditions (1)–(4) of Proposition 6.1 are satisfied for the choice of $c = 0$ and of $M_k = L_1 + \cdots + L_k$ for each $k = 1, \dots, n$. So its conclusion applies for the given c'. Consider the resulting n-system $\mathbf{R} = (R_1, \dots, R_n)$ on $[0,\infty)$. On $[t_{n-1}, t_n]$, the union of the graphs of $R_1, \dots, R_n$ consists of $n-1$ horizontal line segments of ordinates $c', 2c', \dots, (n-1)c'$ and one line segment of slope 1 joining $(t_{n-1}, 0)$ to (t_n, nc'). Since $q_0 = t_{n-1} + c'/2$, we deduce that $\mathbf{R}(q_0) = (c'/2, c', 2c', \dots, (n-1)c')$ and that R_1 has slope 1 on $[q_0, q_0 + c'/2]$. Finally, since $\mathbf{R}$ is rigid of mesh c' on $[t_n,\infty)$, it is also rigid of mesh $c'/2$ on $[q_0,\infty)$. □

This corollary will be useful when it comes to approximate an n-system $\mathbf{L}$ by the map $\mathbf{L}_{\boldsymbol{\xi}}$ attached to a non-zero point $\boldsymbol{\xi} \in K_w^n$, because it reduces the problem to approximating an n-system $\mathbf{R}$ as in the corollary. The property that R_1 has slope 1 to the right of q_0 will simplify the argument.

7. From points to n-systems

The goal of this section is to prove the first and last assertions of Theorem A. To this end, we fix an integer $n \ge 2$, a place $w \in M(K)$, and a non-zero point $\boldsymbol{\xi} \in K_w^n$. As $\mathbf{L}_{\boldsymbol{\xi}}$ and $\mathbf{L}^*_{\boldsymbol{\xi}}$ depend only on the line $\langle \boldsymbol{\xi} \rangle_{K_w}$ spanned by $\boldsymbol{\xi}$ in K_w^n, we assume, to simplify the computations, that

$$\| \boldsymbol{\xi} \|_w = 1.$$

Using the general strategy of Schmidt and Summerer in [22], we will show that the components $L_1, \dots, L_n$ of $\mathbf{L}_{\boldsymbol{\xi}}$ satisfy the hypotheses of Proposition 6.1 for some choice of functions $M_1, \dots, M_n$ and some constant $c = c(K, w, n) \ge 1$. This will ensure the existence of an n-system $\mathbf{P}\colon [0,\infty) \to \mathbb{R}^n$ for which the difference $\mathbf{L}_{\boldsymbol{\xi}} - \mathbf{P}$ is a bounded, and we will show that this is equivalent to $\mathbf{L}^*_{\boldsymbol{\xi}} - \mathbf{P}^*$ being bounded. The precise argument given below is adapted from [17, §2]. In all estimates, the implicit constants involved in the symbol $\asymp$ depend only on K, w and n.

For each $k \in \{1, \dots, n-1\}$, there is a unique bilinear map

$$\begin{array}{rcl} K_w^n \times \bigwedge^k K_w^n & \longrightarrow & \bigwedge^{k-1} K_w^n \\ (\mathbf{y}, \mathbf{X}) & \longmapsto & \mathbf{y} \lrcorner \mathbf{X} \end{array}$$

called *contraction* which satisfies

$$\mathbf{y} \lrcorner (\mathbf{x}_1 \wedge \cdots \wedge \mathbf{x}_k) = \sum_{i=1}^{k} (-1)^{i-1} (\mathbf{y} \cdot \mathbf{x}_i) \mathbf{x}_1 \wedge \cdots \wedge \widehat{\mathbf{x}_i} \wedge \cdots \wedge \mathbf{x}_k \tag{7.1}$$

for any $\mathbf{y}, \mathbf{x}_1, \dots, \mathbf{x}_k \in K_w^n$. For $k = 1$, this is simply the dot product $\mathbf{y} \lrcorner \mathbf{x} = \mathbf{y} \cdot \mathbf{x}$. We use this to define a map $\mathbf{L}^{(k)}_{\boldsymbol{\xi}}$ as follows.

Definition 7.1. Let $k \in \{1, \dots, n-1\}$ and let $N = \binom{n}{k}$. For each non-zero $\mathbf{X} \in \bigwedge^k K^n$, we set

$$D_{\boldsymbol{\xi}}(\mathbf{X}) = \|\, \boldsymbol{\xi} \lrcorner \mathbf{X} \|_w^{d_w/d} \prod_{v \neq w} \|\, \mathbf{X} \|_v^{d_v/d}.$$

As $\|\, \boldsymbol{\xi} \|_w = 1$, this agrees with the definition of section 2.4 for $k = 1$. We also define a map $L_{\boldsymbol{\xi}}(\mathbf{X}, \cdot) \colon [0, \infty) \to \mathbb{R}$ by

$$L_{\boldsymbol{\xi}}(\mathbf{X}, q) = \max\left\{ \log H(\mathbf{X}),\ q + \log D_{\boldsymbol{\xi}}(\mathbf{X}) \right\} \quad (q \geq 0). \tag{7.2}$$

For each $j = 1, \dots, N$ and $q \geq 0$, we denote by $L^{(k)}_{\boldsymbol{\xi},j}(q)$ the smallest real number $t \geq 0$ for which there exists at least j linearly independent elements $\mathbf{X}$ of $\bigwedge^k K^n$ for which $L_{\boldsymbol{\xi}}(\mathbf{X}, q) \leq t$ or equivalently for which

$$H(\mathbf{X}) \leq e^t \quad \text{and} \quad D_{\boldsymbol{\xi}}(\mathbf{X}) \leq e^{t-q}.$$

Finally, we define $\mathbf{L}^{(k)}_{\boldsymbol{\xi}} \colon [0, \infty) \to \mathbb{R}^N$ by $\mathbf{L}^{(k)}_{\boldsymbol{\xi}}(q) = \big(L^{(k)}_{\boldsymbol{\xi},1}(q), \dots, L^{(k)}_{\boldsymbol{\xi},N}(q)\big)$ for each $q \geq 0$.

Since, for a non-zero $\mathbf{X} \in \bigwedge^k K^n$, the numbers $H(\mathbf{X})$ and $D_{\boldsymbol{\xi}}(\mathbf{X})$ depend only on the class of $\mathbf{X}$ in projective space on $\bigwedge^k K^n$, and since for each $B \geq 1$ there are finitely many classes of height at most B, each number $L^{(k)}_{\boldsymbol{\xi},j}(q)$ can indeed be defined as a minimum. For $k = 1$, we recover $\mathbf{L}^{(1)}_{\boldsymbol{\xi}} = \mathbf{L}_{\boldsymbol{\xi}}$. The first step is to compare these maps with the minima of the following families of adelic convex bodies.

Definition 7.2. Let k and N be as in Definition 7.1. For each $q \geq 0$, we denote by $\mathcal{C}^{(k)}_{\boldsymbol{\xi}}(q)$ the convex body of $\bigwedge^k K^n_{\mathbb{A}}$ which consists of the points $\mathbf{X} = (\mathbf{X}_v)$ satisfying

$$\|\, \mathbf{X}_v \|_v \leq 1 \ \text{ for each } v \in M(K) \text{ and } \ \|\, \boldsymbol{\xi} \lrcorner \mathbf{X}_w \|_w \leq e^{-qd/d_w}.$$

We also set $\mathcal{C}_{\boldsymbol{\xi}}(q) = \mathcal{C}^{(1)}_{\boldsymbol{\xi}}(q)$.

Thus $\mathcal{C}_{\boldsymbol{\xi}}(q)$ consists of the points $(\mathbf{x}_v) \in K^n_{\mathbb{A}}$ satisfying

$$\|\, \mathbf{x}_v \|_v \leq 1 \ \text{ for each } v \in M(K) \ \text{ and } \ |\mathbf{x}_w \cdot \boldsymbol{\xi}|_w \leq e^{-qd/d_w}.$$

Its volume is $\mu(\mathcal{C}_{\boldsymbol{\xi}}(q)) \asymp e^{-qd}$. Applying Definition 5.1, we first obtain the following estimate.

Lemma 7.3. *Let $k \in \{1, \dots, n-1\}$, let $\mathbf{X} \in \bigwedge^k K^n \setminus \{0\}$ and $q \geq 0$. Then, we have*

$$\lambda(\mathbf{X}, \mathcal{C}^{(k)}_{\boldsymbol{\xi}}(q)) \asymp \max\{H(\mathbf{X}), e^q D_{\boldsymbol{\xi}}(\mathbf{X})\} = \exp(L_{\boldsymbol{\xi}}(\mathbf{X}, q)).$$

Proof. An idèle $\boldsymbol{a} = (a_v)$ of K of smallest module such that $\mathbf{X} \in \boldsymbol{a}\mathcal{C}^{(k)}_{\boldsymbol{\xi}}(q)$ has $|a_v|_v = \|\, \mathbf{X} \|_v$ for each place $v \neq w$ and $|a_w|_w = \max\{\|\, \mathbf{X} \|_w,\, r \|\, \boldsymbol{\xi} \lrcorner \mathbf{X} \|_w\}$ where r is the smallest element of the valuation group $|K_w^*|_w$ at w with $r \geq e^{qd/d_w}$. The estimate follows since $r \asymp e^{qd/d_w}$ and we have $\lambda(\mathbf{X}, \mathcal{C}^{(k)}_{\boldsymbol{\xi}}(q)) = |\boldsymbol{a}|_{\mathbb{A}}$ for such $\boldsymbol{a}$. $\square$

Lemma 7.4. *Let $k \in \{1, \dots, n-1\}$ and $N = \binom{n}{k}$. For each $q \geq 0$, we have*

$$0 \leq L^{(k)}_{\boldsymbol{\xi},1}(q) \leq \cdots \leq L^{(k)}_{\boldsymbol{\xi},N}(q) \leq q \quad \text{and} \quad \exp(L^{(k)}_{\boldsymbol{\xi},j}(q)) \asymp \lambda_j(\mathcal{C}^{(k)}_{\boldsymbol{\xi}}(q)) \quad (1 \leq j \leq N). \tag{7.3}$$

Moreover, the functions $L^{(k)}_{\boldsymbol{\xi},j}$ are continuous and piecewise linear with slopes 0 and 1 on $[0, \infty)$. Finally, if $L^{(k)}_{\boldsymbol{\xi},1}$ changes slope from 1 to 0 at a point $q > 0$, then $L^{(k)}_{\boldsymbol{\xi},1}(q) = L^{(k)}_{\boldsymbol{\xi},2}(q)$.

Proof. For given $q \geq 0$ and $j \in \{1, \dots, N\}$, the number $L^{(k)}_{\boldsymbol{\xi},j}(q)$ (resp. $\lambda^{\mathbb{A}}_j(\mathcal{C}^{(k)}_{\boldsymbol{\xi}}(q))$) is, by definition, the minimum of

$$\max_{\mathbf{X}\in E} \exp(L_{\boldsymbol{\xi}}(\mathbf{X}, q)) \quad \Big(\text{ resp. } \max_{\mathbf{X}\in E} \lambda(\mathbf{X}, \mathcal{C}^{(k)}_{\boldsymbol{\xi}}(q)) \Big) \tag{7.4}$$

where E runs through all sets of j linearly independent elements of $\bigwedge^k K^n$. Taking for E a set of j products of the form $\mathbf{e}_{i_1} \wedge \cdots \wedge \mathbf{e}_{i_k}$ with $1 \leq i_1 < \cdots < i_k \leq n$, where $(\mathbf{e}_1, \dots, \mathbf{e}_n)$ is the canonical basis of K^n, we deduce that $L^{(k)}_{\boldsymbol{\xi},j}(q) \leq q$ because those products $\mathbf{e}$ have $D_{\boldsymbol{\xi}}(\mathbf{e}) \leq 1 = H(\mathbf{e})$ as $\|\, \boldsymbol{\xi} \,\|_w = 1$. This yields the first set of inequalities in (7.3). Using Lemma 7.3, we also deduce that $\exp(L^{(k)}_{\boldsymbol{\xi},j}(q)) \asymp \lambda^{\mathbb{A}}_j(\mathcal{C}^{(k)}_{\boldsymbol{\xi}}(q))$. The second set of estimates in (7.3) then follows using Proposition 5.3 with $\bigwedge^k K^n_{\mathbb{A}}$ identified to $K^N_{\mathbb{A}}$.

Fix $Q > 0$. For each $q \in [0, Q]$, we have $L^{(k)}_{\boldsymbol{\xi},j}(q) \leq q \leq Q$. Thus in computing $L^{(k)}_{\boldsymbol{\xi},j}(q)$ on $[0, Q]$ in terms of the projective invariants (7.4), it suffices to choose E inside a set F of representatives in $\bigwedge^k K^n$ of points of $\mathbb{P}(\bigwedge^k K^n)$ of height at most e^Q. Since F is finite, we deduce that $L^{(k)}_{\boldsymbol{\xi},j}$ is continuous and piecewise linear with slopes 0 and 1 on $[0, Q]$. As Q can be taken arbitrarily large, this property extends to $[0, \infty)$. Finally, if $L^{(k)}_{\boldsymbol{\xi},1}$ changes slope from 1 to 0 at a point $q > 0$, there exist $\epsilon > 0$ and $\mathbf{X}, \mathbf{Y} \in \bigwedge^k K^n$ such that

$$L^{(k)}_{\boldsymbol{\xi},1}(t) = \begin{cases} L_{\boldsymbol{\xi}}(\mathbf{X}, t) = t + \log D_{\boldsymbol{\xi}}(\mathbf{X}) & \text{for } q - \epsilon \leq t \leq q, \\ L_{\boldsymbol{\xi}}(\mathbf{Y}, t) = \log H(\mathbf{Y}) & \text{for } q \leq t \leq q + \epsilon. \end{cases}$$

Thus $\mathbf{X}, \mathbf{Y}$ are linearly independent and so $L^{(k)}_{\boldsymbol{\xi},1}(q) = L^{(k)}_{\boldsymbol{\xi},2}(q)$. □

The next lemma compares the convex body $\mathcal{C}^{(k)}_{\boldsymbol{\xi}}(q)$ with the k-th compound of $\mathcal{C}_{\boldsymbol{\xi}}(q)$.

Lemma 7.5. *Let k and N be as in Lemma 7.4. For each $q \geq 0$, we have*

$$\textstyle\bigwedge^k \mathcal{C}_{\boldsymbol{\xi}}(q) \subseteq k\mathcal{C}^{(k)}_{\boldsymbol{\xi}}(q) \quad \text{and} \quad \mathcal{C}^{(k)}_{\boldsymbol{\xi}}(q) \subseteq N \bigwedge^k \mathcal{C}_{\boldsymbol{\xi}}(q). \tag{7.5}$$

Proof. Fix a choice of $q \geq 0$ and, for simplicity, set

$$\mathcal{C} := \mathcal{C}_{\boldsymbol{\xi}}(q) = \prod_v \mathcal{C}_v \quad \text{and} \quad \mathcal{C}^{(k)} := \mathcal{C}^{(k)}_{\boldsymbol{\xi}}(q) = \prod_v \mathcal{C}^{(k)}_v.$$

Let $v \in M(K)$ and let $\mathbf{X}_v = \mathbf{x}_1 \wedge \cdots \wedge \mathbf{x}_k$ with $\mathbf{x}_1, \dots, \mathbf{x}_k \in \mathcal{C}_v$. We find

$$\|\, \mathbf{X}_v \,\|_v \leq \|\, \mathbf{x}_1 \,\|_v \cdots \|\, \mathbf{x}_k \,\|_v \leq 1,$$

thus $\mathbf{X}_v \in \mathcal{C}^{(k)}_v$ if $v \neq w$. If $v = w$, the formula (7.1) also yields $\|\, \boldsymbol{\xi} \lrcorner \mathbf{X}_w \,\|_w \leq k^\delta e^{-qd/d_w}$ where $\delta = 1$ if $w \mid \infty$ and $\delta = 0$ else, thus $\mathbf{X}_w \in k\mathcal{C}^{(k)}_w$. This implies the first inclusion in (7.5).

To prove the second inclusion, it suffices to show that each $\mathbf{X}_v \in \mathcal{C}^{(k)}_v$ can be written as a sum of N products of k elements of $\mathcal{C}_v$. For $v \neq w$, this is immediate since $\mathcal{C}_v$ contains $\mathbf{e}_1, \dots, \mathbf{e}_n$ and since the products $\mathbf{e}_{i_1} \wedge \cdots \wedge \mathbf{e}_{i_k}$ with $1 \leq i_1 < \cdots < i_k \leq n$ form an orthonormal basis of K^n_v. As $\|\, \mathbf{X}_v \,\|_v \leq 1$, the N coordinates of $\mathbf{X}_v$ in this basis have absolute values at most one, and we are done. For $v = w$, we complete $\mathbf{u}_1 = \boldsymbol{\xi}$ into an orthonormal basis $(\mathbf{u}_1, \dots, \mathbf{u}_n)$ of K^n_w and we form its dual basis $(\mathbf{u}^*_1, \dots, \mathbf{u}^*_n)$ with respect to the dot product. By Lemma 3.2, this new basis is also orthonormal. Moreover we have $\boldsymbol{\xi} \cdot \mathbf{u}^*_1 = 1$

and $\boldsymbol{\xi}\cdot\mathbf{u}_i^*=0$ for $i=2,\dots,n$. Thus $\mathcal{C}_w$ contains $\mathbf{u}_2^*,\dots,\mathbf{u}_n^*$ as well as $c\mathbf{u}_1^*$ for any $c\in K_w$ with $|c|_w\le e^{-qd/d_w}$. Upon writing

$$\mathbf{X}_w=\sum_{1\le i_1<\cdots<i_k\le n} c_{i_1,\dots,i_k}\mathbf{u}_{i_1}^*\wedge\cdots\wedge\mathbf{u}_{i_k}^*,$$

we find

$$\boldsymbol{\xi}\lrcorner\mathbf{X}_w=\sum_{1<i_2<\cdots<i_k\le n} c_{1,i_2,\dots,i_k}\mathbf{u}_{i_2}^*\wedge\cdots\wedge\mathbf{u}_{i_k}^*.$$

As $\|\mathbf{X}_w\|_w\le 1$ and $\|\boldsymbol{\xi}\lrcorner\mathbf{X}_w\|_w\le e^{-qd/d_w}$, we deduce that $|c_{i_1,\dots,i_k}|_w$ is bounded above by e^{-qd/d_w} if $i_1=1$, and by 1 otherwise, so we are done. □

We can now prove the following part of Theorem A.

Proposition 7.6. *There exists an n-system $\mathbf{P}\colon[0,\infty)\to\mathbb{R}^n$ such that $\|\mathbf{L}_{\boldsymbol{\xi}}-\mathbf{P}\|$ is uniformly bounded by a constant depending only on K, w and n.*

Proof. Set $M_k=L_{\boldsymbol{\xi},1}^{(k)}$ for $k=1,\dots,n-1$ and define $M_n(q)=q$ for each $q\ge 0$. It suffices to show that these functions and the functions $L_k=L_{\boldsymbol{\xi},k}$ satisfy all the hypotheses of Proposition 6.1 for some constant $c=c(K,w,n)$. By Lemma 7.4, we only have to verify the conditions (2) and (4) of that proposition. Theorem 4.1 gives

$$\lambda_1(\mathcal{C}_{\boldsymbol{\xi}}(q))\cdots\lambda_n(\mathcal{C}_{\boldsymbol{\xi}}(q))\asymp\mu(\mathcal{C}_{\boldsymbol{\xi}}(q))^{-1/d}\asymp e^q, \tag{7.6}$$

while for each $k=1,\dots,n-1$, Theorem 4.2 provides

$$\begin{aligned}\lambda_1\big(\textstyle\bigwedge^k\mathcal{C}_{\boldsymbol{\xi}}(q)\big)&\asymp\lambda_1(\mathcal{C}_{\boldsymbol{\xi}}(q))\cdots\lambda_k(\mathcal{C}_{\boldsymbol{\xi}}(q)),\\ \lambda_2\big(\textstyle\bigwedge^k\mathcal{C}_{\boldsymbol{\xi}}(q)\big)&\asymp\lambda_1(\mathcal{C}_{\boldsymbol{\xi}}(q))\cdots\lambda_{k-1}(\mathcal{C}_{\boldsymbol{\xi}}(q))\lambda_{k+1}(\mathcal{C}_{\boldsymbol{\xi}}(q)).\end{aligned}$$

The inclusions of Lemma 7.5 combined with the estimates (7.3) also imply that

$$\lambda_j\big(\textstyle\bigwedge^k\mathcal{C}_{\boldsymbol{\xi}}(q)\big)\asymp\lambda_j\big(\mathcal{C}_{\boldsymbol{\xi}}^{(k)}(q)\big)\asymp\exp(L_{\boldsymbol{\xi},j}^{(k)}(q))$$

for $j=1,\dots,\binom{n}{k}$. Taking logarithms, we obtain the inequalities (2) of Proposition 6.1 as well as

$$|L_{\boldsymbol{\xi},2}^{(k)}(q)-L_1(q)-\cdots-L_{k-1}(q)-L_{k+1}(q)|\le c\quad(1\le k<n,\ q\ge 0),$$

for some constant $c=c(K,w,n)$. Finally, if, for some $k<n$, the function M_k changes slope from 1 to 0 at a point $q>0$, Lemma 7.4 gives $L_{\boldsymbol{\xi},2}^{(k)}(q)=L_{\boldsymbol{\xi},1}^{(k)}(q)=M_k(q)$. Then comparing the last estimate with that of Proposition 6.1 (2), we obtain $|L_{k+1}(q)-L_k(q)|\le 2c$. Thus the condition (4) of Proposition 6.1 holds as well. □

To compare approximation to $\mathbf{L}_{\boldsymbol{\xi}}$ by an n-system $\mathbf{P}$ and approximation to $\mathbf{L}_{\boldsymbol{\xi}}^*$ by the dual map $\mathbf{P}^*$ defined by (2.4), we first note the following equality.

Lemma 7.7. *We have $\mathbf{L}_{\boldsymbol{\xi}}^{(n-1)}=\mathbf{L}_{\boldsymbol{\xi}}^*$.*

Proof. Consider the K-linear map $\varphi\colon K^n \to \bigwedge^{n-1} K^n$ given by

$$\varphi(\mathbf{x}) = \mathbf{x} \lrcorner (\mathbf{e}_1 \wedge \cdots \wedge \mathbf{e}_n)$$

for each $\mathbf{x} \in K^n$. Writing $\mathbf{x} = (x_1, \dots, x_n)$, we find that

$$\varphi(\mathbf{x}) = \sum_{i=1}^{n} (-1)^{i-1} x_i \mathbf{e}_1 \wedge \cdots \wedge \widehat{\mathbf{e}_i} \wedge \cdots \wedge \mathbf{e}_n.$$

Thus φ is an isomorphism and, for each place v of K, it extends to a K_v-linear isometry $\varphi_v\colon K_v^n \to \bigwedge^{n-1} K_v^n$ given by the same formulas. Moreover, a short computation shows that

$$\| \boldsymbol{\xi} \lrcorner \varphi_w(\mathbf{x}) \|_w = \| \mathbf{x} \wedge \boldsymbol{\xi} \|_w$$

for each $\mathbf{x} \in K_w^n$. Thus, for each non-zero $\mathbf{x} \in K^n$, we have

$$H(\varphi(\mathbf{x})) = H(\mathbf{x}) \quad \text{and} \quad D_{\boldsymbol{\xi}}(\varphi(\mathbf{x})) = D^*_{\boldsymbol{\xi}}(\mathbf{x}),$$

and the conclusion follows since φ is an isomorphism. □

Lemma 7.8. *There is a constant $c^* = c^*(K, w, n)$ such that $|L^*_{\boldsymbol{\xi},j}(q) + L_{\boldsymbol{\xi},k}(q) - q| \le c^*$ for each $q \ge 0$ and each $j, k \in \{1, \dots, n\}$ with $j + k = n + 1$.*

Proof. For q, j and k as above, Theorem 4.2 combined with (7.6) provides

$$\lambda_j\big(\textstyle\bigwedge^{n-1}\mathcal{C}_{\boldsymbol{\xi}}(q)\big) \asymp \lambda_1(\mathcal{C}_{\boldsymbol{\xi}}(q)) \cdots \widehat{\lambda_k(\mathcal{C}_{\boldsymbol{\xi}}(q))} \cdots \lambda_n(\mathcal{C}_{\boldsymbol{\xi}}(q)) \asymp \frac{e^q}{\lambda_k(\mathcal{C}_{\boldsymbol{\xi}}(q))}.$$

Using Lemmas 7.5, 7.4 and 7.7 in this order, we also find

$$\lambda_j\big(\textstyle\bigwedge^{n-1}\mathcal{C}_{\boldsymbol{\xi}}(q)\big) \asymp \lambda_j(\mathcal{C}^{(n-1)}_{\boldsymbol{\xi}}(q)) \asymp \exp(L^{(n-1)}_{\boldsymbol{\xi},j}(q)) = \exp(L^*_{\boldsymbol{\xi},j}(q)),$$

while $\lambda_k(\mathcal{C}_{\boldsymbol{\xi}}(q)) \asymp \exp(L_{\boldsymbol{\xi},k}(q))$. The conclusion follows by taking logarithms. □

We deduce the following complement to Proposition 7.6.

Proposition 7.9. *Let $\mathbf{P} = (P_1, \dots, P_n)$ be an n-system on $[0, \infty)$. If one of the conditions (2.5) from Theorem A holds with a constant c, then the other holds with c replaced by $c + c^*\sqrt{n}$ where c^* comes from Lemma 7.8.*

Proof. With q, j and k as in Lemma 7.8, the definition of $\mathbf{P}^*$ in (2.4) gives $P^*_j(q) + P_k(q) = q$. Hence the inequality of the lemma may be restated as

$$\big|(L^*_{\boldsymbol{\xi},j}(q) - P^*_j(q)) + (L_{\boldsymbol{\xi},k}(q) - P_k(q))\big| \le c^*,$$

and the result follows. □

We conclude this section with the following observation.

Lemma 7.10. *For $q \ge 0$, define $\mathcal{C}^*_{\boldsymbol{\xi}}(q)$ to be the set of points $(\mathbf{x}_v) \in K^n_{\mathbb{A}}$ satisfying*

$$\| \mathbf{x}_v \|_v \le 1 \quad \text{for each } v \in M(K) \quad \text{and} \quad \| \mathbf{x}_w \wedge \boldsymbol{\xi} \|_w \le e^{-qd/d_w}.$$

Then $\lambda_j(\mathcal{C}^*_{\boldsymbol{\xi}}(q)) = \lambda_j(\mathcal{C}^{(n-1)}_{\boldsymbol{\xi}}(q)) \asymp \exp(L^{(n-1)}_j(q)) = \exp(L^*_j(q))$ for each $j = 1, \dots, n$.

Proof. Going back to the proof of Lemma 7.7, we find that the $K_{\mathbb{A}}$-linear isomorphism $\varphi_{\mathbb{A}}\colon K_{\mathbb{A}}^n \to \bigwedge^{n-1} K_{\mathbb{A}}^n$ which extends φ maps $\mathcal{C}_{\boldsymbol{\xi}}^*(q)$ to $\mathcal{C}_{\boldsymbol{\xi}}^{(n-1)}(q)$ for each $q \geq 0$. Thus these convex bodies have the same minima. The remaining estimates follow from (7.3) in Lemma 7.4, and from Lemma 7.7. □

8. Construction of bases

As in the preceding section, we assume $n \geq 2$, and we fix a place $w \in M(K)$. We also set

$$S = M_\infty(K) \cup \{w\}$$

and denote by $\mathcal{O}_S = \cap_{v \notin S}(K \cap \mathcal{O}_v)$ the ring of S-integers of K. The goal of this section is to provide a general recursive construction of bases of $\mathcal{O}_S^n$ as an $\mathcal{O}_S$-module. In the next section, we will use it to complete the proof of Theorem A. The general strategy is similar to that of [17, §5] but complicated by the fact that we need these bases to obey several properties at each place of S. In particular, we will need them to be almost orthogonal in K_v^n for each $v \in S \setminus \{w\}$, in the sense of section 3.4 for $L = K_v$. We start by recalling two general results of approximation by elements of $\mathcal{O}_S$ within $\prod_{v\in S} K_v$.

The group of S-units of K is the group $\mathcal{O}_S^*$ of invertible elements of $\mathcal{O}_S$. It is well-known that its image under the logarithmic embedding, namely the set of points $(\log|\varepsilon|_v)_{v\in S}$ with $\varepsilon \in \mathcal{O}_S^*$, forms a lattice within the hyperplane of $\mathbb{R}^S$ of points $(x_v)_{v\in S}$ with $\sum_{v\in S} d_v x_v = 0$. Thus, there is a constant $c_2 = c_2(K, S) \geq 1$ with the following property.

Lemma 8.1. *For any choice of positive real numbers $(r_v)_{v\in S}$ with $\prod_{v\in S} r_v^{d_v} = 1$, there exists an S-unit $\varepsilon \in \mathcal{O}_S^*$ which satisfies $c_2^{-1} r_v \leq |\varepsilon|_v \leq c_2 r_v$ for each $v \in S$.*

From [10, Theorem 3] of Mahler, there is also a constant $c_3 = c_3(K, S) \geq 1$ with the following property.

Lemma 8.2. *For any $(a_v)_{v\in S} \in \prod_{v\in S} K_v$ and any family of positive real numbers $(t_v)_{v\in S}$ with $\prod_{v\in S} t_v^{d_v/d} \geq c_3$, there exists an S-integer $\alpha \in \mathcal{O}_S$ which satisfies $|\alpha - a_v|_v \leq t_v$ for each $v \in S$.*

In fact, the result of Mahler shows this with a constant c_3 that depends only on K when each t_v belongs to the valuation group of K_v. As we do not require this, our constant c_3 depends on S as well, but in a weak form.

From now on, we fix a constant C in the valuation group of K_w, with

$$C \geq n2^{n+1}(c_3c_4)^{d/d_w} \quad \text{where} \quad c_4 = n2^n(2ec_2)^2. \tag{8.1}$$

In concordance with the notation of section 3.4 for $L = K_w$, we set

$$\delta = \begin{cases} 1 & \text{if } w \mid \infty, \\ 0 & \text{otherwise.} \end{cases}$$

For the other places $v \in S\setminus\{w\}$, no special notation is needed since they all are archimedean and so the results of that section apply to $L = K_v$ with $\delta = 1$. For convenience, we also define

$$\Delta = \{(a_1,\dots,a_n) \in \mathbb{Z}^n\,;\, 0 \le a_1 \le \cdots \le a_n\}. \tag{8.2}$$

We are interested in bases of $\mathcal{O}_S^n$ with three kinds of properties.

Definition 8.3. Let $\mathbf{x} = (\mathbf{x}_1,\dots,\mathbf{x}_n)$ be a basis of $\mathcal{O}_S^n$ over $\mathcal{O}_S$. We say that

- $\mathbf{x}$ is *admissible* if, for any $v \in S\setminus\{w\}$, it is almost orthogonal in K_v^n and satisfies $1 \le \|\mathbf{x}_j\|_v \le (2ec_2)^2$ for $j = 1,\dots,n$;
- $\mathbf{x}$ has *size* $\boldsymbol{a} = (a_1,\dots,a_n) \in \Delta$ if $C^{a_j} \le \|\mathbf{x}_j\|_w \le (1+\delta)C^{a_j}$ for $j = 1,\dots,n$;
- $\mathbf{x}$ has *type* (k,ℓ) for integers $1 \le k < \ell \le n$ if

$$\mathrm{dist}_w\big(\mathbf{x}_\ell, \langle \mathbf{x}_1,\dots,\widehat{\mathbf{x}_k},\dots,\mathbf{x}_{\ell-1}\rangle_{K_w}\big) \ge 1 - \frac{1}{2^{\ell-1}}.$$

We start by two quick consequences.

Lemma 8.4. *Suppose that $\mathbf{x} = (\mathbf{x}_1,\dots,\mathbf{x}_n)$ is an admissible basis of $\mathcal{O}_S^n$ over $\mathcal{O}_S$ of size $\boldsymbol{a} = (a_1,\dots,a_n) \in \Delta$. Then, for $j = 1,\dots,\ell$, we have $C^{a_j d_w/d} \le H(\mathbf{x}_j) \le c_5 C^{a_j d_w/d}$ where $c_5 = (2ec_2)^2$.*

Proof. Let $j \in \{1,\dots,n\}$. For each place v of K not in S, the n-tuple $\mathbf{x}$ is a basis of $\mathcal{O}_v^n$ over $\mathcal{O}_v$, hence $\|\mathbf{x}_j\|_v = 1$. Thus we have $H(\mathbf{x}_j) = \prod_{v\in S} \|\mathbf{x}_j\|_v^{d_v/d}$ and the conclusion follows. $\square$

Lemma 8.5. *Let k, ℓ, m be integers with $1 \le k < \ell \le m \le n$ and let $\mathbf{y} = (\mathbf{y}_1,\dots,\mathbf{y}_n)$ be an admissible basis of $\mathcal{O}_S^n$ over $\mathcal{O}_S$. Suppose that the subsequences $(\mathbf{y}_1,\dots,\widehat{\mathbf{y}_\ell},\dots,\mathbf{y}_m)$ and $(\mathbf{y}_1,\dots,\widehat{\mathbf{y}_k},\dots,\mathbf{y}_m)$ are both almost orthogonal in K_w^n. Then, the subspaces*

$$V_1 = \langle \mathbf{y}_1,\dots,\widehat{\mathbf{y}_\ell},\dots,\mathbf{y}_m\rangle_{K_w} \quad \text{and} \quad V_2 = \langle \mathbf{y}_1,\dots,\widehat{\mathbf{y}_k},\dots,\mathbf{y}_m\rangle_{K_w}$$

that they span in K_w^n satisfy

$$\mathrm{dist}_w(V_1,V_2)^{d_w/d} \le e^4 \frac{H(\langle \mathbf{y}_1,\dots,\mathbf{y}_m\rangle_K)}{H(\mathbf{y}_1)\cdots H(\mathbf{y}_m)}.$$

Proof. Using Lemma 3.11 with $L = K_w$, we find

$$\mathrm{dist}_w(V_1,V_2)^{d_w/d} \le \left(e^{4\delta}\frac{\|\mathbf{y}_1\wedge\cdots\wedge\mathbf{y}_m\|_w}{\|\mathbf{y}_1\|_w\cdots\|\mathbf{y}_m\|_w}\right)^{d_w/d} = A\frac{H(\langle \mathbf{y}_1,\dots,\mathbf{y}_m\rangle_K)}{H(\mathbf{y}_1)\cdots H(\mathbf{y}_m)},$$

where

$$A = e^{4\delta d_w/d}\prod_{v\in S\setminus\{w\}}\left(\frac{\|\mathbf{y}_1\|_v\cdots\|\mathbf{y}_m\|_v}{\|\mathbf{y}_1\wedge\cdots\wedge\mathbf{y}_m\|_v}\right)^{d_v/d}.$$

Since $\mathbf{y}$ is an admissible basis of $\mathcal{O}_S^n$, the sequence $(\mathbf{y}_1,\dots,\mathbf{y}_m)$ is almost orthogonal in K_v^n for each place $v \in S$ other than w. As such a place v is archimedean, Lemma 3.10 shows that the corresponding factor of A is bounded above by $e^{2d_v/d}$. This gives $A \le e^4$. $\square$

The main result of this section is the following construction which, in essence, generalizes [17, Lemma 5.1]. The crucial novelty is the introduction of an S-unit ε in the condition (3) below (in the context of [17] where $K=\mathbb{Q}$ and $S=\{\infty\}$, it would simply be $\varepsilon=\pm1$).

Lemma 8.6. *Let h, k, ℓ be integers with*

$$1 \le k < \ell \le n \quad \text{and} \quad 1 \le h \le \ell \le n.$$

Suppose that elements $\boldsymbol{a}=(a_1,\dots,a_n)$ and $\boldsymbol{b}=(b_1,\dots,b_n)$ of Δ satisfy

$$(1) \qquad b_\ell > a_\ell \quad \text{and} \quad (b_1,\dots,\widehat{b_\ell},\dots,b_n)=(a_1,\dots,\widehat{a_h},\dots,a_n).$$

Suppose moreover that $\mathbf{x}=(\mathbf{x}_1,\dots,\mathbf{x}_n)$ is an admissible basis of $\mathcal{O}_S^n$ over $\mathcal{O}_S$ of size $\boldsymbol{a}$. Then there exists an admissible basis $\mathbf{y}=(\mathbf{y}_1,\dots,\mathbf{y}_n)$ of $\mathcal{O}_S^n$ over $\mathcal{O}_S$ of size $\boldsymbol{b}$ and type (k,ℓ) such that

$$(2) \qquad (\mathbf{y}_1,\dots,\widehat{\mathbf{y}_\ell},\dots,\mathbf{y}_n)=(\mathbf{x}_1,\dots,\widehat{\mathbf{x}_h},\dots,\mathbf{x}_n),$$

$$(3) \qquad \mathbf{y}_\ell \in \varepsilon\mathbf{x}_h + \langle \mathbf{x}_1,\dots,\widehat{\mathbf{x}_h},\dots,\mathbf{x}_\ell\rangle_{\mathcal{O}_S} \text{ for some } \varepsilon\in\mathcal{O}_S^*.$$

Proof. We use (2) as a definition of $\mathbf{y}_1,\dots,\widehat{\mathbf{y}_\ell},\dots,\mathbf{y}_n$. Then $\mathbf{y}=(\mathbf{y}_1,\dots,\mathbf{y}_n)$ is a basis of $\mathcal{O}_S^n$ over $\mathcal{O}_S$ for any choice of $\mathbf{y}_\ell$ satisfying (3).

We set $r_v = 2e^2c_2\|\,\mathbf{x}_h\|_v^{-1}$ for each $v\in S\setminus\{w\}$, and define r_w by the condition $\prod_{v\in S} r_v^{d_v}=1$. Since $\mathbf{x}$ is admissible, we have $r_v \ge (2c_2)^{-1}$ for $v\ne w$, so $r_w^{d_w}\le (2c_2)^d$. Then Lemma 8.1 provides an S-unit $\varepsilon\in\mathcal{O}_S^*$ with

$$(8.3) \qquad 2e^2 \le \|\,\varepsilon\mathbf{x}_h\|_v \le 2e^2c_2^2 \quad \text{for each } v\in S\setminus\{w\},$$

$$(8.4) \qquad |\varepsilon|_w^{d_w} \le (c_2 r_w)^{d_w} \le (2c_2)^{2d}.$$

For each $v\in S$, we define

$$U_v = \langle \mathbf{x}_1,\dots,\mathbf{x}_\ell\rangle_{K_v},$$
$$V_v = \langle \mathbf{x}_1,\dots,\widehat{\mathbf{x}_h},\dots,\mathbf{x}_\ell\rangle_{K_v} = \langle \mathbf{y}_1,\dots,\mathbf{y}_{\ell-1}\rangle_{K_v},$$

and we choose a unit vector $\mathbf{u}_v\in U_v$ such that

$$U_v = \langle \mathbf{u}_v\rangle_{K_v} \perp_{\mathrm{top}} V_v.$$

If $v\ne w$, we write

$$(8.5) \qquad \varepsilon\mathbf{x}_h = c_v\mathbf{u}_v + \sum_{j=1}^{\ell-1} c_{v,j}\mathbf{y}_j$$

with coefficients c_v and $c_{v,j}$ in K_v. For $v=w$, we also define

$$W_w = \langle \mathbf{y}_1,\dots,\widehat{\mathbf{y}_k},\dots,\mathbf{y}_{\ell-1}\rangle_{K_w},$$

and choose a unit vector $\mathbf{v}_w\in V_w$ such that

$$V_w = \langle \mathbf{v}_w\rangle_{K_w} \perp_{\mathrm{top}} W_w.$$

This provides a decomposition $U_w = \langle \mathbf{u}_w\rangle_{K_w} \perp_{\mathrm{top}} \langle \mathbf{v}_w\rangle_{K_w} \perp_{\mathrm{top}} W_w$. We choose $B\in K_w$ with

$$(8.6) \qquad |B|_w = (1+\delta/2)C^{b_\ell}.$$

This is possible because if $w \nmid \infty$, then $\delta = 0$ and C belongs to the valuation group of K_w. We then write

$$\varepsilon \mathbf{x}_h = c_w \mathbf{u}_w + B\mathbf{v}_w + \sum_{j=1}^{\ell-1} c_{w,j}\mathbf{y}_j \tag{8.7}$$

with c_w and $c_{w,j}$ in K_w. The approximation lemma 8.2 provides, for each $j = 1, \dots, \ell-1$, an S-integer $\alpha_j \in \mathcal{O}_S$ such that

$$\begin{aligned} &|\alpha_j - c_{v,j}|_v \le c_4^{-1} \quad \text{for all } v \in S \setminus \{w\},\\ &|\alpha_j - c_{w,j}|_w \le (c_3c_4)^{d/d_w} \end{aligned} \tag{8.8}$$

for the constant $c_4 = n2^n(2ec_2)^2$ defined in (8.1). Then the point

$$\mathbf{y}_\ell = \varepsilon \mathbf{x}_h - \sum_{j=1}^{\ell-1} \alpha_j \mathbf{y}_j \tag{8.9}$$

fulfills the condition (3) and so $\mathbf{y} = (\mathbf{y}_1, \dots, \mathbf{y}_n)$ is an $\mathcal{O}_S$-basis of $\mathcal{O}_S^n$.

1° To show that $\mathbf{y}$ is admissible, we fix a place $v \in S \setminus \{w\} \subseteq M_\infty(K)$. Since $\mathbf{x}$ is admissible, we obtain directly $1 \le \|\mathbf{y}_j\|_v \le (2ec_2)^2$ for each $j \neq \ell$ because of the equality (2). As $\mathbf{x}$ is almost orthogonal in K_v^n, its subsequence

$$(\mathbf{x}_1, \dots, \widehat{\mathbf{x}_h}, \dots, \mathbf{x}_\ell) = (\mathbf{y}_1, \dots, \mathbf{y}_{\ell-1})$$

is also almost orthogonal. Moreover, for each integer j with $\ell + 1 \le j \le n$, we have $\mathbf{x}_j = \mathbf{y}_j$ and $\langle \mathbf{x}_1, \dots, \mathbf{x}_{j-1}\rangle_{K_v} = \langle \mathbf{y}_1, \dots, \mathbf{y}_{j-1}\rangle_{K_v}$, thus

$$\mathrm{dist}_v\big(\mathbf{y}_j,\ \langle \mathbf{y}_1, \dots, \mathbf{y}_{j-1}\rangle_{K_v}\big) = \mathrm{dist}_v\big(\mathbf{x}_j,\ \langle \mathbf{x}_1, \dots, \mathbf{x}_{j-1}\rangle_{K_v}\big) \ge 1 - (1/2)^{j-1}.$$

Since $\langle \mathbf{y}_1, \dots, \mathbf{y}_{\ell-1}\rangle_{K_v} = V_v$, it only remains to show that

$$1 \le \|\mathbf{y}_\ell\|_v \le (2ec_2)^2 \quad \text{and} \quad \mathrm{dist}_v(\mathbf{y}_\ell, V_v) \ge 1 - (1/2)^{\ell-1}. \tag{8.10}$$

To this end, we first note that, since $(\mathbf{x}_1, \dots, \mathbf{x}_n)$ is almost orthogonal in K_v^n, Lemmas 3.6 and 3.10 yield

$$1 \ge \mathrm{dist}_v(\mathbf{x}_h, V_v) = \frac{\|\mathbf{x}_1 \wedge \cdots \wedge \mathbf{x}_\ell\|_v}{\|\mathbf{x}_h\|_v \|\mathbf{x}_1 \wedge \cdots \wedge \widehat{\mathbf{x}_h} \wedge \cdots \wedge \mathbf{x}_\ell\|_v} \ge \frac{\|\mathbf{x}_1 \wedge \cdots \wedge \mathbf{x}_\ell\|_v}{\|\mathbf{x}_1\|_v \cdots \|\mathbf{x}_\ell\|_v} \ge e^{-2}.$$

On the other hand, Lemma 3.5 applied to the decomposition (8.5) gives

$$\mathrm{dist}_v(\mathbf{x}_h, V_v) = \mathrm{dist}_v(\varepsilon\mathbf{x}_h, V_v) = \frac{|c_v|_v}{\|\varepsilon\mathbf{x}_h\|_v}.$$

Using the estimates (8.3) for $\|\varepsilon\mathbf{x}_h\|_v$, we deduce that

$$2 \le |c_v|_v \le 2e^2c_2^2. \tag{8.11}$$

Combining (8.5) and (8.9), we obtain

$$\mathbf{y}_\ell = c_v\mathbf{u}_v + \sum_{j=1}^{\ell-1}(c_{v,j} - \alpha_j)\mathbf{y}_j.$$

Using (8.8), this decomposition of $\mathbf{y}_\ell$ implies

$$\|\mathbf{y}_\ell - c_v\mathbf{u}_v\|_v \le \sum_{j=1}^{\ell-1} c_4^{-1}\|\mathbf{y}_j\|_v \le nc_4^{-1}(2ec_2)^2 = 2^{-n},$$

and thus $1 \le \|\mathbf{y}_\ell\|_v \le (2ec_2)^2$ by (8.11). Finally Lemma 3.5 gives

$$\mathrm{dist}_v(\mathbf{y}_\ell, V_v) = \frac{|c_v|_v}{\|\mathbf{y}_\ell\|_v} \ge \frac{|c_v|_v}{|c_v|_v + 2^{-n}} \ge \frac{2}{2+2^{-n}} \ge 1 - \frac{1}{2^{n+1}},$$

which completes the proof of (8.10). Thus $\mathbf{y}$ is admissible.

2° We now show that $\mathbf{y}$ has size $\boldsymbol{b}$. Since $\mathbf{x}$ has size $\boldsymbol{a}$, the relations (1) and (2) reduce the problem to showing that

$$C^{b_\ell} \le \|\mathbf{y}_\ell\|_w \le (1+\delta)C^{b_\ell}. \tag{8.12}$$

To prove this, we first combine (8.7) and (8.9) to obtain

$$\mathbf{y}_\ell = c_w\mathbf{u}_w + B\mathbf{v}_w + \sum_{j=1}^{\ell-1}(c_{w,j} - \alpha_j)\mathbf{y}_j,$$

and note that the decomposition (8.7) implies

$$|c_w|_w = \|c_w\mathbf{u}_w\|_w \le \|\varepsilon\mathbf{x}_h\|_w \le (2c_2)^{2d/d_w}\|\mathbf{x}_h\|_w,$$

where the last estimation comes from (8.4). Using (8.8), we deduce that

$$\|\mathbf{y}_\ell - B\mathbf{v}_w\|_w \le |c_w|_w + \sum_{j=1}^{\ell-1}(c_3c_4)^{d/d_w}\|\mathbf{y}_j\|_w \le (c_3c_4)^{d/d_w}\sum_{j=1}^{\ell}\|\mathbf{x}_j\|_w.$$

We also note that $\|\mathbf{x}_j\|_w \le 2C^{a_j} \le 2C^{b_\ell-1}$ for $j = 1,\dots,\ell$ because $\mathbf{x}$ has size $\boldsymbol{a}$ and the hypothesis (1) gives $a_1 \le \cdots \le a_\ell < b_\ell$. Using the hypothesis (8.1) on C, we conclude that

$$\|\mathbf{y}_\ell - B\mathbf{v}_w\|_w \le 2n(c_3c_4)^{d/d_w}C^{b_\ell-1} \le 2^{-n}C^{b_\ell}. \tag{8.13}$$

Together with the formula (8.6) for $|B|_w$, this yields (8.12). So $\mathbf{y}$ has size $\boldsymbol{b}$.

3° It remains to show that $\mathbf{y}$ has type (k,ℓ).

By (8.13), we have a decomposition $\mathbf{y}_\ell = B\mathbf{v}_w + \mathbf{z}$ with $\|\mathbf{z}\|_w \le 2^{-n}|B|_w$. Thus,

$$\mathrm{dist}_w(\mathbf{y}_\ell, W_w) \ge \frac{|B|_w - \|\mathbf{z}\|_w}{\|\mathbf{y}_\ell\|_w} \ge \frac{1-2^{-n}}{1+2^{-n}} \ge 1 - \frac{1}{2^{n-1}} \qquad \text{if } w \mid \infty,$$

$$\mathrm{dist}_w(\mathbf{y}_\ell, W_w) \ge \frac{|B|_w}{\|\mathbf{y}_\ell\|_w} = 1 \qquad \text{otherwise.}$$

In both cases, this yields $\mathrm{dist}_w(\mathbf{y}_\ell, W_w) \ge 1 - \delta/2^{\ell-1}$. So $\mathbf{y}$ has type (k,ℓ). ∎

We will also need the following complementary result.

Lemma 8.7. *With the same hypotheses and notation, let m be an integer with $\ell \le m \le n$.*

(i) *We have $\langle \mathbf{x}_1,\dots,\mathbf{x}_m\rangle_K = \langle \mathbf{y}_1,\dots,\mathbf{y}_m\rangle_K$. If $m < n$, we also have $\mathbf{x}_{m+1} = \mathbf{y}_{m+1}$.*

(ii) *If* $(\mathbf{x}_1,\dots,\widehat{\mathbf{x}_h},\dots,\mathbf{x}_\ell)$ *is almost orthogonal in* K_w^n*, then* $(\mathbf{y}_1,\dots,\widehat{\mathbf{y}_k},\dots,\mathbf{y}_\ell)$ *is as well almost orthogonal in* K_w^n*.*

(iii) *If both* $(\mathbf{x}_1,\dots,\widehat{\mathbf{x}_h},\dots,\mathbf{x}_m)$ *and* $(\mathbf{y}_1,\dots,\widehat{\mathbf{y}_k},\dots,\mathbf{y}_m)$ *are almost orthogonal in* K_w^n*, then the subspaces of* K_w^n *that they span,*

$$V_1=\langle \mathbf{x}_1,\dots,\widehat{\mathbf{x}_h},\dots,\mathbf{x}_m\rangle_{K_w} \quad \textit{and} \quad V_2=\langle \mathbf{y}_1,\dots,\widehat{\mathbf{y}_k},\dots,\mathbf{y}_m\rangle_{K_w}$$

satisfy

$$\operatorname{dist}_w(V_1,V_2)^{d_w/d}\le e^4\frac{H(\langle\mathbf{y}_1,\dots,\mathbf{y}_m\rangle_K)}{H(\mathbf{y}_1)\cdots H(\mathbf{y}_m)}\le e^4\frac{c_5}{C^{d_w/d}}\frac{H(\langle\mathbf{x}_1,\dots,\mathbf{x}_m\rangle_K)}{H(\mathbf{x}_1)\cdots H(\mathbf{x}_m)}. \tag{8.14}$$

When (iii) holds for some $m<n$, the estimates (8.14) together with Corollary 3.8 allow us to connect the distances from $\mathbf{x}_{m+1}=\mathbf{y}_{m+1}$ to V_1 and V_2 in terms of heights only.

Proof. Part (i) follows immediately from the conditions (2) and (3) of Lemma 8.6.

To prove (ii), we note that $(\mathbf{x}_1,\dots,\widehat{\mathbf{x}_h},\dots,\mathbf{x}_\ell)=(\mathbf{y}_1,\dots,\mathbf{y}_{\ell-1})$. Thus, if this sequence is almost orthogonal in K_w^n, so is its subsequence $(\mathbf{y}_1,\dots,\widehat{\mathbf{y}_k},\dots,\mathbf{y}_{\ell-1})$. Since $\mathbf{y}$ is of type (k,ℓ), we then conclude that $(\mathbf{y}_1,\dots,\widehat{\mathbf{y}_k},\dots,\mathbf{y}_\ell)$ is almost orthogonal in K_w^n.

Under the hypotheses of (iii), the first inequality in (8.14) follows from Lemma 8.5 because $(\mathbf{x}_1,\dots,\widehat{\mathbf{x}_h},\dots,\mathbf{x}_m)$ coincides with $(\mathbf{y}_1,\dots,\widehat{\mathbf{y}}_\ell,\dots,\mathbf{y}_m)$. To prove the second inequality, we use the fact that $\mathbf{x}$ and $\mathbf{y}$ have respective size $\boldsymbol{a}$ and $\boldsymbol{b}$ with $a_h\le a_\ell<b_\ell$. By Lemma 8.4, we obtain

$$\frac{H(\mathbf{x}_1)\cdots H(\mathbf{x}_m)}{H(\mathbf{y}_1)\cdots H(\mathbf{y}_m)}=\frac{H(\mathbf{x}_h)}{H(\mathbf{y}_\ell)}\le c_5C^{(a_h-b_\ell)d_w/d}\le\frac{c_5}{C^{d_w/d}}.$$

The required inequality follows since $H(\langle\mathbf{y}_1,\dots,\mathbf{y}_m\rangle_K)=H(\langle\mathbf{x}_1,\dots,\mathbf{x}_m\rangle_K)$ by (i). ☐

We conclude with the following existence result which for us replaces [17, Lemma 5.2].

Lemma 8.8. *Let* $\boldsymbol{a}=(a_1,\dots,a_n)\in\mathbb{Z}^n$ *with* $0\le a_1<\cdots<a_n$*. There exists an admissible basis* $\mathbf{x}=(\mathbf{x}_1,\dots,\mathbf{x}_n)$ *of* $\mathcal{O}_S^n$ *over* $\mathcal{O}_S$ *of size* $\boldsymbol{a}$ *and type* $(1,n)$ *such that* $(\mathbf{x}_1,\dots,\mathbf{x}_{n-1})$ *is almost orthogonal in* K_w^n*.*

Proof. Starting from the canonical basis $(\mathbf{e}_1,\dots,\mathbf{e}_n)$ of K^n, Lemma 8.6 applied recursively n times with $h=k=1$ and $\ell=n$ provides points $\mathbf{x}_1,\dots,\mathbf{x}_n$ of $\mathcal{O}_S^n$ such that, for each $j=0,\dots,n$, the n-tuple $(\mathbf{e}_{j+1},\dots,\mathbf{e}_n,\mathbf{x}_1,\dots,\mathbf{x}_j)$ is an admissible basis of $\mathcal{O}_S^n$ of size $(0,\dots,0,a_1,\dots,a_j)$ and type $(1,n)$. For each j with $2\le j\le n-1$, we find

$$\operatorname{dist}_w\left(\mathbf{x}_j,\langle\mathbf{x}_1,\dots,\mathbf{x}_{j-1}\rangle_{K_w}\right)\ge\operatorname{dist}_w\left(\mathbf{x}_j,\langle\mathbf{e}_{j+2},\dots,\mathbf{e}_n,\mathbf{x}_1,\dots,\mathbf{x}_{j-1}\rangle_{K_w}\right)\ge 1-1/2^{n-1}.$$

Thus $(\mathbf{x}_1,\dots,\mathbf{x}_{n-1})$ is almost orthogonal in K_w^n. ☐

9. From n-systems to points

Let $\mathbf{L}\colon [0,\infty) \to \mathbb{R}^n$ be an arbitrary n-system. To complete the proof of Theorem A, it remains to show the existence of a non-zero point $\boldsymbol{\xi} \in K_w^n$ for which $\|\mathbf{L} - \mathbf{L}_{\boldsymbol{\xi}}\|$ is bounded above by a constant depending only on K, w and n. To this end, consider the n-system $\mathbf{R} = (R_1, \dots, R_n)$ provided by Corollary 6.2 for the choice of $c' = 2c$, where

$$c = \frac{d_w}{d}\log(C)$$

for the constant $C = C(K, w, n)$ of the preceding section, satisfying (8.1). Since $\|\mathbf{L} - \mathbf{R}\|$ is bounded above by a constant depending only on K, n and w, it suffices to construct a non-zero point $\boldsymbol{\xi} \in K_w^n$ for which $\|\mathbf{R} - \mathbf{L}_{\boldsymbol{\xi}}\|$ is also bounded above by such a constant.

Let $q_0 = (n^2 - 2n + 1)c$, so that the restriction of $\mathbf{R}$ to $[q_0, \infty)$ is rigid of mesh c. We denote by $(q_i)_{0 \le i < s}$ the finite or infinite sequence of switch points of $\mathbf{R}$ on that interval, with cardinality $s \in \{1, 2, \dots\} \cup \{\infty\}$. For each integer i with $0 \le i < s$, we set

$$\boldsymbol{a}^{(i)} = c^{-1}\mathbf{R}(q_i) = (a_1^{(i)}, \dots, a_n^{(i)}) \in \Delta,$$

where $\Delta \subset \mathbb{Z}^n$ is defined by (8.2). We have $R_j(q_i) = c a_j^{(i)}$ for $j = 1, \dots, n$ and

$$q_i = c a_1^{(i)} + \cdots + c a_n^{(i)} \quad (0 \le i < s). \tag{9.1}$$

We also denote by k_i the index j for which the right derivative of R_j at q_i is 1 and, when $i > 0$, we denote by ℓ_i the index j for which the left derivative of R_j at q_i is 1. By the choice of $\mathbf{R}$, we have $k_0 = 1$. Finally, we set $\ell_0 = n$. Then, for each integer i with $1 \le i < s$, we have

(P1) $1 = k_0 < \ell_0 = n$ and $1 \le k_i < \ell_i \le n$,

(P2) $\ell_i \ge k_{i-1}$ and $a_{\ell_i}^{(i)} > a_{\ell_i}^{(i-1)}$,

(P3) $\big(a_1^{(i)}, \dots, \widehat{a_{\ell_i}^{(i)}}, \dots, a_n^{(i)}\big) = \big(a_1^{(i-1)}, \dots, \widehat{a_{k_{i-1}}^{(i-1)}}, \dots, a_n^{(i-1)}\big)$.

From these data, it is a simple matter to reconstruct the function $\mathbf{R}$. Let

$$\Phi\colon \mathbb{R}^n \to \{(x_1, \dots, x_n) \in \mathbb{R}^n\,;\, x_1 \le x_2 \le \cdots \le x_n\}$$

denote the continuous function which reorders the coordinates of a point as a monotonically increasing sequence, and set $q_s = \infty$ if $s < \infty$. Then, we have

$$\begin{aligned}\mathbf{R}(q) &= \Phi(\mathbf{R}(q_i) + (q - q_i)\mathbf{e}_{k_i})\\ &= \Phi(R_1(q_i), \dots, R_{k_i}(q_i) + q - q_i, \dots, R_n(q_i))\end{aligned} \quad \text{for each } q \in [q_i, q_{i+1}). \tag{9.2}$$

The formula also extends to $q = q_{i+1}$ if $i + 1 < s$.

We first apply the results of the preceding section to construct a specific sequence of bases of $\mathcal{O}_S^n$. Its relevance to our problem will become clear in the corollaries that we derive afterwards.

Proposition 9.1. *There exists a sequence $(\mathbf{x}^{(i)})_{0\le i<s}$ of bases of $\mathcal{O}_S^n$ over $\mathcal{O}_S$ such that, for each integer i with $0 \le i < s$, the basis $\mathbf{x}^{(i)} = (\mathbf{x}_1^{(i)}, \dots, \mathbf{x}_n^{(i)})$ is admissible of size $\boldsymbol{a}^{(i)}$ and type (k_i, ℓ_i) with the additional property that, when $i \ge 1$,*

$$(1)\qquad \big(\mathbf{x}_1^{(i)}, \dots, \widehat{\mathbf{x}_{\ell_i}^{(i)}}, \dots, \mathbf{x}_n^{(i)}\big) = \big(\mathbf{x}_1^{(i-1)}, \dots, \widehat{\mathbf{x}_{k_{i-1}}^{(i-1)}}, \dots, \mathbf{x}_n^{(i-1)}\big),$$

$$(2)\qquad \mathbf{x}_{\ell_i}^{(i)} \in \varepsilon_i \mathbf{x}_{k_{i-1}}^{(i-1)} + \langle \mathbf{x}_1^{(i-1)}, \dots, \widehat{\mathbf{x}_{k_{i-1}}^{(i-1)}}, \dots, \mathbf{x}_{\ell_i}^{(i-1)} \rangle_{\mathcal{O}_S} \text{ for some } S\text{-unit } \varepsilon_i \in \mathcal{O}_S^*.$$

We may further require that the sequence $\widehat{\mathbf{x}}^{(-1)} := (\mathbf{x}_1^{(0)}, \dots, \mathbf{x}_{n-1}^{(0)})$ is almost orthogonal in K_w^n. Then, for each integer i with $0 \le i < s$, the sequence $\widehat{\mathbf{x}}^{(i)} := (\mathbf{x}_1^{(i)}, \dots, \widehat{\mathbf{x}_{k_i}^{(i)}}, \dots, \mathbf{x}_n^{(i)})$ is also almost orthogonal in K_w^n. Finally, for each i with $-1 \le i < s$, choose a unit vector $\mathbf{u}_i$ of K_w^n which is orthogonal to each vector of $\widehat{\mathbf{x}}^{(i)}$ with respect to the dot product. Then we further have

$$(9.3)\qquad \mathrm{dist}_w(\mathbf{u}_i, \mathbf{u}_j) \le 2^\delta \exp((4 - q_{i+1})d/d_w) \quad (-1 \le i < j < s).$$

Proof. Lemma 8.6 provides recursively such a sequence of bases starting from any admissible basis $\mathbf{x}^{(0)} = (\mathbf{x}_1^{(0)}, \dots, \mathbf{x}_n^{(0)})$ of size $\boldsymbol{a}^{(0)}$ and type (k_0, ℓ_0). To build $\mathbf{x}^{(i)}$ from $\mathbf{x}^{(i-1)}$ for an integer i with $1 \le i < s$, we apply this lemma with $h = k_{i-1}$, $(k, \ell) = (k_i, \ell_i)$, $\boldsymbol{a} = \boldsymbol{a}^{(i-1)}$ and $\boldsymbol{b} = \boldsymbol{a}^{(i)}$. The hypotheses of the lemma are fulfilled by virtue of the conditions (P1)–(P3).

By Lemma 8.8, we may choose the initial basis $\mathbf{x}^{(0)}$ so that $\widehat{\mathbf{x}}^{(-1)}$ is almost orthogonal in K_w^n. Assuming this, we now prove by induction that $\widehat{\mathbf{x}}^{(i)}$ is almost orthogonal in K_w^n for each i with $0 \le i < s$.

We first note that $\widehat{\mathbf{x}}^{(0)} = (\mathbf{x}_2^{(0)}, \dots, \mathbf{x}_n^{(0)})$ is almost orthogonal in K_w^n because $\mathbf{x}^{(0)}$ has type $(1, n)$ and the sequence $(\mathbf{x}_2^{(0)}, \dots, \mathbf{x}_{n-1}^{(0)})$ is almost orthogonal in K_w^n, as a subsequence of the almost orthogonal sequence $\widehat{\mathbf{x}}^{(-1)}$.

Suppose now that $\widehat{\mathbf{x}}^{(i)}$ is almost orthogonal in K_w^n for each $i = 0, \dots, t-1$ where t is an integer with $1 \le t < s$. To complete the induction step, we will show, by induction on m, that $(\mathbf{x}_1^{(t)}, \dots, \widehat{\mathbf{x}_{k_t}^{(t)}}, \dots, \mathbf{x}_m^{(t)})$ is almost orthogonal in K_w^n for each $m = \ell_t, \dots, n$. For $m = \ell_t$, this follows from Lemma 8.7 (ii) since $(\mathbf{x}_1^{(t-1)}, \dots, \widehat{\mathbf{x}_{k_{t-1}}^{(t-1)}}, \dots, \mathbf{x}_{\ell_t}^{(t-1)})$ is almost orthogonal in K_w^n. If $\ell_t = n$, we are done. Otherwise, let m be an integer with $\ell_t \le m < n$ for which $(\mathbf{x}_1^{(t)}, \dots, \widehat{\mathbf{x}_{k_t}^{(t)}}, \dots, \mathbf{x}_m^{(t)})$ is almost orthogonal in K_w^n. Since $\ell_0 = n > m$, there is a largest integer r with $0 \le r < t$ such that $\ell_r > m$. This means that $\ell_{r+1}, \dots, \ell_t \le m$ and so $k_r, \dots, k_t \le m$ by (P1) and (P2). Moreover, we have $\mathbf{x}_{m+1}^{(r)} = \dots = \mathbf{x}_{m+1}^{(t)}$ by Lemma 8.7 (i). Define

$$(9.4)\qquad U^{(i)} = \langle \mathbf{x}_1^{(i)}, \dots, \mathbf{x}_m^{(i)} \rangle_{K_w} \quad \text{and} \quad V^{(i)} = \langle \mathbf{x}_1^{(i)}, \dots, \widehat{\mathbf{x}_{k_i}^{(i)}}, \dots, \mathbf{x}_m^{(i)} \rangle_{K_w}$$

for $i = r, \dots, t$. We claim that

$$(9.5)\qquad \mathrm{dist}_w(\mathbf{x}_{m+1}^{(r)}, V^{(r)}) \ge 1 - \frac{\delta}{2^m} \quad \text{and} \quad \mathrm{dist}_w(V^{(r)}, V^{(t)}) \le \frac{1}{2^m}.$$

If we take this for granted, then Corollary 3.8 gives $\mathrm{dist}_w(\mathbf{x}^{(t)}_{m+1}, V^{(t)}) \geq 1 - \delta/2^{m-1}$ which is exactly what we need to complete the induction on m, and thus to complete the main induction as well.

If $r \geq 1$ and $m+1 < \ell_r$, the $(m+1)$-tuple $(\mathbf{x}^{(r)}_1, \dots, \mathbf{x}^{(r)}_{m+1})$ is almost orthogonal in K^n_w as a subsequence of

$$\big(\mathbf{x}^{(r)}_1, \dots, \widehat{\mathbf{x}^{(r)}_{\ell_r}}, \dots, \mathbf{x}^{(r)}_n\big) = \big(\mathbf{x}^{(r-1)}_1, \dots, \widehat{\mathbf{x}^{(r-1)}_{k_{r-1}}}, \dots, \mathbf{x}^{(r-1)}_n\big).$$

If $r = 0$ and $m+1 < \ell_0 = n$, it is also almost orthogonal in K^n_w, as a subsequence of $\widehat{\mathbf{x}}^{(-1)}$. So, independently of r, if $m+1 < \ell_r$, we obtain

$$\mathrm{dist}_w(\mathbf{x}^{(r)}_{m+1}, V^{(r)}) \geq \mathrm{dist}_w(\mathbf{x}^{(r)}_{m+1}, U^{(r)}) \geq 1 - \frac{\delta}{2^m},$$

which gives the first inequality in (9.5). If $m+1 = \ell_r$, the latter inequality holds by construction, since $\mathbf{x}^{(r)}$ has type (k_r, ℓ_r).

To prove the second inequality in (9.5), we apply Lemma 8.7 (iii). For $i = r+1, \dots, t$, it gives

$$\mathrm{dist}_w(V^{(i-1)}, V^{(i)})^{d_w/d} \leq e^4 \frac{H(\langle \mathbf{x}^{(i)}_1, \dots, \mathbf{x}^{(i)}_m \rangle_K)}{H(\mathbf{x}^{(i)}_1) \cdots H(\mathbf{x}^{(i)}_m)} \leq e^4 \frac{c_5}{C^{d_w/d}} \frac{H(\langle \mathbf{x}^{(i-1)}_1, \dots, \mathbf{x}^{(i-1)}_m \rangle_K)}{H(\mathbf{x}^{(i-1)}_1) \cdots H(\mathbf{x}^{(i-1)}_m)},$$

and so, for the same values of i, we obtain

$$\mathrm{dist}_w(V^{(i-1)}, V^{(i)})^{d_w/d} \leq e^4 \left(\frac{c_5}{C^{d_w/d}}\right)^{i-r} \frac{H(\langle \mathbf{x}^{(r)}_1, \dots, \mathbf{x}^{(r)}_m \rangle_K)}{H(\mathbf{x}^{(r)}_1) \cdots H(\mathbf{x}^{(r)}_m)} \leq e^4 \left(\frac{c_5}{C^{d_w/d}}\right)^{i-r}.$$

Since $C \geq 2^n (e^4 c_5)^{d/d_w}$, this yields $\mathrm{dist}_w(V^{(i-1)}, V^{(i)}) \leq 2^{-(i-r)n}$ for $i = r+1, \dots, n$ and so by the triangle inequality of Lemma 3.4 we obtain

$$\mathrm{dist}_w(V^{(r)}, V^{(t)}) \leq \sum_{i=r+1}^{t} \mathrm{dist}_w(V^{(i-1)}, V^{(i)}) \leq \frac{1}{2^{n-1}} \leq \frac{1}{2^m},$$

which completes the proof of (9.5).

By the above, the sequence $\widehat{\mathbf{x}}^{(i)}$ is almost orthogonal in K^n_w for each i with $-1 \leq i < s$. For those i, take $V^{(i)}$ to be the subspace of K^n_w spanned by $\widehat{\mathbf{x}}^{(i)}$ so that $(V^{(i)})^\perp = \langle \mathbf{u}_i \rangle_{K_w}$. When $0 \leq i < s$, both $\widehat{\mathbf{x}}^{(i-1)}$ and $\widehat{\mathbf{x}}^{(i)}$ are almost orthogonal subsequences of $\mathbf{x}^{(i)}$. Then Lemmas 3.9 and 8.5 yield

$$\mathrm{dist}_w(\mathbf{u}_{i-1}, \mathbf{u}_i)^{d_w/d} = \mathrm{dist}_w(V^{(i-1)}, V^{(i)})^{d_w/d} \leq \frac{e^4}{H(\mathbf{x}^{(i)}_1) \cdots H(\mathbf{x}^{(i)}_n)} \quad (0 \leq i < s)$$

because $\langle \mathbf{x}^{(i)}_1, \dots, \mathbf{x}^{(i)}_n \rangle_K = K^n$ has height 1. Since the basis $\mathbf{x}^{(i)}$ is admissible of size $a^{(i)}$, Lemma 8.4 further gives

$$\log H(\mathbf{x}^{(i)}_j) \geq a^{(i)}_j (d_w/d) \log(C) = c a^{(i)}_j \quad \text{for } j = 1, \dots, n.$$

Using (9.1), we conclude that

$$\mathrm{dist}_w(\mathbf{u}_{i-1}, \mathbf{u}_i)^{d_w/d} \leq \exp\left(4 - \sum_{j=1}^{n} c a^{(i)}_j\right) = \exp(4 - q_i) \quad (0 \leq i < s).$$

Since $(q_i)_{0\le i<s}$ is a strictly increasing sequence of multiples of c and since $cd/d_w = \log(C) \ge \log(2)$, we deduce that

$$\mathrm{dist}_w(\mathbf{u}_{j-1}, \mathbf{u}_j) \le \exp((4-q_j)d/d_w) \le (1/2)^{j-i} \exp((4-q_i)d/d_w)$$

for each pair of integers $0 \le i < j < s$. Then, (9.3) follows from the triangle inequality of Lemma 3.4. □

For the corollary below, we recall our convention that $q_s = \infty$ when $s < \infty$. As a special case of (7.2), we also recall that, for each $\mathbf{x} \in K^n$, each non-zero $\boldsymbol{\xi} \in K_w^n$ and each $q \ge 0$, we have by definition

$$L_{\boldsymbol{\xi}}(\mathbf{x}, q) = \max\{\log H(\mathbf{x}), q + \log D_{\boldsymbol{\xi}}(\mathbf{x})\}.$$

Corollary 9.2. *Under the hypotheses of Proposition 9.1, there is a unit vector $\boldsymbol{\xi} \in K_w^n$ such that, for each integer i with $0 \le i < s$ and each $q \in [q_i, q_{i+1})$, we have*

$$L_{\boldsymbol{\xi}}(\mathbf{x}_j^{(i)}, q) \le c_6 + R_j(q_i) + \begin{cases} q - q_i & \text{if } j = k_i, \\ 0 & \text{otherwise,} \end{cases}$$

where $c_6 = 6 + \log(c_5)$.

Proof. Consider the sequence of unit vectors $(\mathbf{u}_i)_{-1\le i<s}$ given by the proposition. If $s = \infty$, it follows from (9.3) that its image in $\mathbb{P}^{n-1}(K_w)$ converges to the class of a unit vector $\boldsymbol{\xi} \in K_w^n$ such that

$$\mathrm{dist}_w(\mathbf{u}_i, \boldsymbol{\xi}) \le 2^\delta \exp((4-q_{i+1})d/d_w) \quad (-1 \le i < s). \tag{9.6}$$

When $s < \infty$, we have $q_s = \infty$ by convention, and the same holds with $\boldsymbol{\xi} = \mathbf{x}_{s-1}$.

We now fix i and j with $0 \le i < s$ and $1 \le j \le n$. For each $q \ge 0$, we have

$$L_{\boldsymbol{\xi}}(\mathbf{x}_j^{(i)}, q) = \log H(\mathbf{x}_j^{(i)}) + \max\left\{0,\ q + \frac{d_w}{d}\log\frac{|\mathbf{x}_j^{(i)}\cdot\boldsymbol{\xi}|_w}{\|\,\mathbf{x}_j^{(i)}\,\|_w}\right\}. \tag{9.7}$$

We also have $\mathbf{x}_{k_i}^{(i)} \cdot \mathbf{u}_{i-1} = 0$ because $\mathbf{x}_{k_i}^{(i)}$ belongs to the sequence $\widehat{\mathbf{x}}^{(i-1)}$. So (3.7) yields

$$|\mathbf{x}_{k_i}^{(i)}\cdot\boldsymbol{\xi}|_w \le 2^\delta \|\,\mathbf{x}_{k_i}^{(i)}\,\|_w\, \mathrm{dist}_w(\mathbf{u}_{i-1}, \boldsymbol{\xi}).$$

If $j \ne k_i$, we have instead $\mathbf{x}_j^{(i)} \cdot \mathbf{u}_i = 0$ because $\mathbf{x}_j^{(i)}$ belongs to $\widehat{\mathbf{x}}^{(i)}$, and so (3.7) yields

$$|\mathbf{x}_j^{(i)}\cdot\boldsymbol{\xi}|_w \le 2^\delta \|\,\mathbf{x}_j^{(i)}\,\|_w\, \mathrm{dist}_w(\mathbf{u}_i, \boldsymbol{\xi}).$$

Using (9.6) and noting that $4^\delta \le \exp(2d/d_w)$, we deduce that

$$\frac{d_w}{d}\log\frac{|\mathbf{x}_j^{(i)}\cdot\boldsymbol{\xi}|_w}{\|\,\mathbf{x}_j^{(i)}\,\|_w} \le \begin{cases} 6 - q_i & \text{if } j = k_i, \\ 6 - q_{i+1} & \text{otherwise.} \end{cases}$$

Since $\mathbf{x}^{(i)}$ has size $\boldsymbol{a}$, Lemma 8.4 gives $\log H(\mathbf{x}_j^{(i)}) \le \log(c_5) + ca_j^{(i)} = c_6 - 6 + R_j(q_i)$. The conclusion follows by substituting the last two estimates into (9.7). □

We can now complete the proof of Theorem A as follows.

Corollary 9.3. *Let $\boldsymbol{\xi}$ be as in Corollary 9.2. There is a constant $c_7 = c_7(K, w, n)$ such that*

$$\max_{1\le j\le n} |L_{\boldsymbol{\xi},j}(q) - R_j(q)| \le c_7 \tag{9.8}$$

for each $q \ge 0$.

Proof. Fix an integer i with $0 \le i < s$ and a point $q \in [q_i, q_{i+1})$. Denote by

$$(r_1, \dots, r_n) = \Phi\big(L_{\boldsymbol{\xi}}(\mathbf{x}_1^{(i)}, q), \dots, L_{\boldsymbol{\xi}}(\mathbf{x}_n^{(i)}, q)\big)$$

the numbers $L_{\boldsymbol{\xi}}(\mathbf{x}_j^{(i)}, q)$ with $1 \le j \le n$ written in monotonically increasing order. Since $\mathbf{x}^{(i)}$ is a basis of $\mathcal{O}_S^n$ over $\mathcal{O}_S$, it is also a basis of K^n over K, and so by definition we have

$$L_{\boldsymbol{\xi},j}(q) \le r_j \quad (1 \le j \le n).$$

By Corollary 9.2 and formula (9.2), we also have

$$r_j \le c_6 + R_j(q) \quad (1 \le j \le n).$$

By definition of an n-system, we further have $\sum_{j=1}^n R_j(q) = q$. Thus, by Lemma 7.4 (for $k = 1$) and the estimate (7.6), we find

$$\sum_{j=1}^{n} \big(c_6 + R_j(q) - L_{\boldsymbol{\xi},j}(q)\big) = nc_6 + q - \sum_{j=1}^{n} L_{\boldsymbol{\xi},j}(q) \le c_6 + c_7$$

for a constant $c_7 = c_7(K, w, n) \ge \max\{c_6, q_0\}$. This yields (9.8) because each summand in the main sum on the left is non-negative. Finally, (9.8) also holds for each $q \in [0, q_0)$, because for such q, the numbers $R_j(q)$ and $L_{\boldsymbol{\xi},j}(q)$ belong to $[0, q_0) \subseteq [0, c_7)$. □

10. Spectra of exponents of approximation

We fix a place w of K and an integer $n \ge 2$. For each non-zero $\boldsymbol{\xi} \in K_w^n$, we write $\widehat{\omega}(\boldsymbol{\xi})$ for $\widehat{\omega}(\boldsymbol{\xi}, K, w)$ and similarly for the three other exponents introduced in section 2.4. Our goal is to show that their spectrum is independent of the choice of K and w and more precisely that it can be expressed in terms of n-systems, as mentioned in the introduction. We will also generalize to the present setting the exponents of Laurent from [9] and show that the same applies to their spectrum. We start with the following observation.

Lemma 10.1. *For each non-zero $\boldsymbol{\xi} \in K_w^n$, we have*

$$\liminf_{q\to\infty} \frac{L_{\boldsymbol{\xi},1}(q)}{q} = \frac{1}{\omega(\boldsymbol{\xi}) + 1}, \qquad \limsup_{q\to\infty} \frac{L_{\boldsymbol{\xi},1}(q)}{q} = \frac{1}{\widehat{\omega}(\boldsymbol{\xi}) + 1},$$

$$\liminf_{q\to\infty} \frac{L^*_{\boldsymbol{\xi},1}(q)}{q} = \frac{1}{\lambda(\boldsymbol{\xi}) + 1}, \qquad \limsup_{q\to\infty} \frac{L^*_{\boldsymbol{\xi},1}(q)}{q} = \frac{1}{\widehat{\lambda}(\boldsymbol{\xi}) + 1}.$$

Proof. By Definition 2.1, the number $\widehat{\omega}(\boldsymbol{\xi})$ (resp. $\omega(\boldsymbol{\xi})$) is the supremum of all $\omega \ge 0$ such that, for each sufficiently large $t > 0$ (resp. for arbitrarily large $t > 0$), there is a non-zero point $\mathbf{x} \in K^n$ with $H(\mathbf{x}) \le e^t$ and $D_{\boldsymbol{\xi}}(\mathbf{x}) \le e^{-\omega t}$. By definition of $L_{\boldsymbol{\xi},1}$ in section 2.5, the existence of such $\mathbf{x}$ translates into $L_{\boldsymbol{\xi},1}((\omega + 1)t) \le t$. Using the change of variables $q = (\omega + 1)t$, we deduce that $\widehat{\omega}(\boldsymbol{\xi})$ (resp. $\omega(\boldsymbol{\xi})$) is the supremum of all $\omega \ge 0$ for which

$q^{-1}L_{\boldsymbol{\xi},1}(q) \le 1/(\omega+1)$ for each sufficiently large $q>0$ (resp. for arbitrarily large $q>0$). This yields the first row of formulas. The second one is proved in the same way. □

For any function $\mathbf{P}=(P_1,\dots,P_n)\colon [0,\infty)\to\mathbb{R}^n$ and any $j=1,\dots,n$, we define

$$\underline{\varphi}_j(\mathbf{P}) = \liminf_{q\to\infty}\frac{P_j(q)}{q} \quad \text{and} \quad \bar{\varphi}_j(\mathbf{P}) = \limsup_{q\to\infty}\frac{P_j(q)}{q}.$$

The following generalization of [17, Corollary 1.4] characterizes the spectrum of $(\omega,\widehat{\omega},\lambda,\widehat{\lambda})$.

Proposition 10.2. *The set $\mathcal{S}$ of quadruples*

$$\left(\frac{1}{\omega(\boldsymbol{\xi})+1},\frac{1}{\widehat{\omega}(\boldsymbol{\xi})+1},\frac{1}{\lambda(\boldsymbol{\xi})+1},\frac{1}{\widehat{\lambda}(\boldsymbol{\xi})+1}\right)\in[0,1]^4$$

where $\boldsymbol{\xi}\in K_w^n$ has K-linearly independent coordinates coincides with the set of quadruples

$$\big(\underline{\varphi}_1(\mathbf{P}),\bar{\varphi}_1(\mathbf{P}),1-\bar{\varphi}_n(\mathbf{P}),1-\underline{\varphi}_n(\mathbf{P})\big)\in[0,1]^4$$

where $\mathbf{P}=(P_1,\dots,P_n)$ is an n-system with first component P_1 unbounded.

Proof. By the lemma, $\mathcal{S}$ consists of the points $(\underline{\varphi}_1(\mathbf{L}_{\boldsymbol{\xi}}),\bar{\varphi}_1(\mathbf{L}_{\boldsymbol{\xi}}),\underline{\varphi}_1(\mathbf{L}^*_{\boldsymbol{\xi}}),\bar{\varphi}_1(\mathbf{L}^*_{\boldsymbol{\xi}}))$ for all $\boldsymbol{\xi}\in K_w^n$ with K-linearly independent coordinates. By definition of $L_{\boldsymbol{\xi},1}$ in section 2.5, that condition on $\boldsymbol{\xi}$ is equivalent to asking that $L_{\boldsymbol{\xi},1}$ is unbounded. Thus, by Theorem A, the set $\mathcal{S}$ consists of the points $\big(\underline{\varphi}_1(\mathbf{P}),\bar{\varphi}_1(\mathbf{P}),\underline{\varphi}_1(\mathbf{P}^*),\bar{\varphi}_1(\mathbf{P}^*)\big)$ where $\mathbf{P}=(P_1,\dots,P_n)$ is an n-system whose first component P_1 is unbounded. The conclusion follows since, for any n-system $\mathbf{P}$, one has $\underline{\varphi}_1(\mathbf{P}^*)=1-\bar{\varphi}_n(\mathbf{P})$ and $\bar{\varphi}_1(\mathbf{P}^*)=1-\underline{\varphi}_n(\mathbf{P})$. □

Corollary 10.3. *The set $\mathcal{S}$ is independent of the choice of K and w. In particular, since Jarník's identity* (1.3) *holds for any point $\boldsymbol{\xi}$ of $\mathbb{Q}_\infty^3=\mathbb{R}^3$ with $\mathbb{Q}$-linearly independent coordinates, it also holds for any point $\boldsymbol{\xi}$ of K_w^3 with K-linearly independent coordinates.*

To generalize the exponents of Laurent from [9], we fix an integer k with $1\le k\le n-1$, and we assume for simplicity that $\|\boldsymbol{\xi}\|_w=1$. For each non-zero $\mathbf{X}\in\bigwedge^k K^n$, we define

$$D^*_{\boldsymbol{\xi}}(\mathbf{X}) = \|\boldsymbol{\xi}\wedge\mathbf{X}\|_w^{d_w/d}\prod_{v\neq w}\|\mathbf{X}\|_v^{d_v/d}$$

in addition to the quantity $D_{\boldsymbol{\xi}}(\mathbf{X})$ from Definition 7.1. For $k=1$, this agrees with the notation of section 2.4 since $\|\boldsymbol{\xi}\|_w=1$. We note the following fact.

Lemma 10.4. *For given $\omega\ge 0$ and $Q\ge 1$, the following conditions are equivalent:*

(i) *there exists a non-zero $\mathbf{X}\in\bigwedge^k K^n$ with $H(\mathbf{X})\le Q$ and $D^*_{\boldsymbol{\xi}}(\mathbf{X})\le Q^{-\omega}$;*
(ii) *there exists a non-zero $\mathbf{Y}\in\bigwedge^{n-k} K^n$ with $H(\mathbf{Y})\le Q$ and $D_{\boldsymbol{\xi}}(\mathbf{Y})\le Q^{-\omega}$.*

Proof. Consider the K_v-linear isometry $\varphi_{k,v}\colon \bigwedge^k K_v^n \to \bigwedge^{n-k} K_v^n$ from section 3.3 for each $v\in M(K)$. They all restrict to a single K-linear isomorphism $\varphi_k\colon \bigwedge^k K^n \to \bigwedge^{n-k}K^n$. Moreover, for each $\mathbf{X}\in\bigwedge^k K_w^n$, we have

$$\boldsymbol{\xi}\lrcorner\,\varphi_{k,w}(\mathbf{X}) = \varphi_{k+1,w}(\mathbf{X}\wedge\boldsymbol{\xi})$$

(cf. [4, §3, Lemma 2]) and thus $\|\,\boldsymbol{\xi}\lrcorner\varphi_{k,w}(\mathbf{X})\,\|_w = \|\,\mathbf{X}\wedge\boldsymbol{\xi}\,\|_w$. We conclude that, if $\mathbf{X}\in\bigwedge^k K^n$ is non-zero, then the point $\mathbf{Y}=\varphi_k(\mathbf{X})\in\bigwedge^{n-k}K^n$ is non-zero with $H(\mathbf{Y})=H(\mathbf{X})$ and $D_{\boldsymbol{\xi}}(\mathbf{Y})=D^*_{\boldsymbol{\xi}}(\mathbf{X})$. Thus the two conditions are equivalent. □

Definition 10.5. We denote by $\omega_{k-1}(\boldsymbol{\xi})$ (resp. $\widehat{\omega}_{k-1}(\boldsymbol{\xi})$) the supremum of all $\omega\ge 0$ for which the equivalent conditions of Lemma 10.4 are fulfilled for arbitrarily large values of Q (resp. for all sufficiently large values of Q).

For $k=1$, applying condition (i) shows that $\omega_0(\boldsymbol{\xi})=\lambda(\boldsymbol{\xi})$ and $\widehat{\omega}_0(\boldsymbol{\xi})=\widehat{\lambda}(\boldsymbol{\xi})$. For $k=n-1$, applying condition (ii) instead shows that $\omega_{n-2}(\boldsymbol{\xi})=\omega(\boldsymbol{\xi})$ and $\widehat{\omega}_{n-2}(\boldsymbol{\xi})=\widehat{\omega}(\boldsymbol{\xi})$. Moreover, for $K=\mathbb{Q}$, $w=\infty$ and any choice of k, the number $\omega_{k-1}(\boldsymbol{\xi})$ defined above coincides with the exponent of Laurent from [9], in view of condition (i): see [9, §2, Remark] or [4, §4, Proposition].

Arguing as in Lemma 10.1, using condition (ii) and the definition of $L^{(k-1)}_{\boldsymbol{\xi},1}$, we deduce that

$$\frac{1}{\omega_{k-1}(\boldsymbol{\xi})+1}=\liminf_{q\to\infty}\frac{L^{(n-k)}_{\boldsymbol{\xi},1}(q)}{q}\quad\text{and}\quad\frac{1}{\widehat{\omega}_{k-1}(\boldsymbol{\xi})+1}=\limsup_{q\to\infty}\frac{L^{(n-k)}_{\boldsymbol{\xi},1}(q)}{q}.$$

On the other hand, the proof of Proposition 7.6 shows that $L^{(j)}_{\boldsymbol{\xi},1}$ differs by a bounded function from $L_{\boldsymbol{\xi},1}+\cdots+L_{\boldsymbol{\xi},j}$ for each $j=1,\dots,n-1$. We conclude that the spectrum of these $2n-2$ exponents is independent of K and w, and characterized as follows (cf. [18, Proposition 3.1]).

Proposition 10.6. *The set of points*

$$\big((\omega_0(\boldsymbol{\xi})+1)^{-1},\dots,(\omega_{n-2}(\boldsymbol{\xi})+1)^{-1},(\widehat{\omega}_0(\boldsymbol{\xi})+1)^{-1},\dots,(\widehat{\omega}_{n-2}(\boldsymbol{\xi})+1)^{-1}\big)$$

where $\boldsymbol{\xi}\in K_w^n$ has K-linearly independent coordinates coincides with the set of points

$$\big(\underline{\psi}_{n-1}(\mathbf{P}),\dots,\underline{\psi}_1(\mathbf{P}),\bar{\psi}_{n-1}(\mathbf{P}),\dots,\bar{\psi}_1(\mathbf{P})\big)$$

where $\mathbf{P}=(P_1,\dots,P_n)$ is an n-system with first component P_1 unbounded, and

$$\underline{\psi}_j(\mathbf{P})=\liminf_{q\to\infty}\frac{P_1(q)+\cdots+P_j(q)}{q}\quad\text{and}\quad\bar{\psi}_j(\mathbf{P})=\limsup_{q\to\infty}\frac{P_1(q)+\cdots+P_j(q)}{q}$$

for $j=1,\dots,n-1$.

As in [9, Definition 2], the exponent $\omega_{k-1}(\boldsymbol{\xi})$ can be described geometrically as the supremum of all $\omega\ge 0$ for which there are infinitely many subspaces V of K^n of dimension k that satisfy

$$\inf\{\mathrm{dist}_w(\boldsymbol{\xi},\mathbf{x})\,;\,\mathbf{x}\in V\}\le H(V)^{-(\omega+1)d/d_w}.$$

We simply sketch the proof which is similar in spirit to that of [4, §4, Proposition], based on the results of section 7. We first observe that, in defining $\omega_{k-1}(\boldsymbol{\xi})$ through condition (ii) of Lemma 10.4, we may take for $\mathbf{Y}$ a point of K^n which realize the first minimum of $\mathcal{C}^{(n-k)}_{\boldsymbol{\xi}}(t)$ where $t=(\omega+1)\log(Q)$. Since that convex body is comparable to $\bigwedge^{n-k}\mathcal{C}_{\boldsymbol{\xi}}(t)$, we may even take for $\mathbf{Y}$ the wedge product of the first $n-k$ points of a basis of K^n which realize the minima of $\mathcal{C}_{\boldsymbol{\xi}}(t)$ (see the comments after Theorem 4.2). Let W denote the subspace of K^n spanned by these points, so that $\mathbf{Y}$ spans $\bigwedge^{n-k}W$, and let $V=W^{\perp}$. Then, going back to the proof of the lemma, we find that $\varphi_k(\bigwedge^k V) = \bigwedge^{n-k} W$ and so $\mathbf{Y} = \varphi_k(\mathbf{X})$ for some generator $\mathbf{X}$ of $\bigwedge^k V$. In defining $\omega_{k-1}(\boldsymbol{\xi})$ through condition (i), we may thus assume that $\mathbf{X}$ has this form. For such a point, Lemma 3.6 yields $D^*_{\boldsymbol{\xi}}(\mathbf{X}) = \mathrm{dist}_w(\boldsymbol{\xi}, \overline{V})^{d_w/d} H(V)$ where $\overline{V} = \langle V\rangle_{K_w}$ is the topological closure of V in K_w^n, and the claim follows.

11. The principle of Thunder

In his alternative proof of the adelic Minkowski's theorem [24], Jeff Thunder relates the successive minima of a convex body of $K^n_{\mathbb{A}}$ to those of an appropriate convex body of $\mathbb{Q}^{dn}_{\mathbb{A}}$. We formulate his idea below as a general principle.

To this end, we use the following construction where $\mathbb{Q}^{dn}$ and K^n are viewed respectively as subsets of $\mathbb{Q}^{dn}_{\mathbb{A}}$ and $K^n_{\mathbb{A}}$ under the diagonal embedding.

Lemma 11.1. *Let $T\colon \mathbb{Q}^{dn} \to K^n$ be a $\mathbb{Q}$-linear isomorphism. For each place u of $\mathbb{Q}$ and each place v of K above u, we denote by $T_v\colon \mathbb{Q}^{dn}_u \to K_v$ the $\mathbb{Q}_u$-linear map which extends T. Then*

$$\begin{array}{rccl} T_u : & \mathbb{Q}^{dn}_u & \longrightarrow & \prod_{v|u} K^n_v \\ & \mathbf{x} & \longmapsto & (T_v(\mathbf{x}))_{v|u} \end{array}$$

is a $\mathbb{Q}_u$-linear isomorphism. Moreover the map

$$\begin{array}{rccl} T_{\mathbb{A}} : & \mathbb{Q}^{dn}_{\mathbb{A}} & \longrightarrow & K^n_{\mathbb{A}} \\ & (\mathbf{x}_u)_{u\in M(\mathbb{Q})} & \longmapsto & \big((T_v(\mathbf{x}_u))_{v|u}\big)_{u\in M(\mathbb{Q})} \end{array}$$

is the unique $\mathbb{Q}_{\mathbb{A}}$-linear map which extends T. It is also an isomorphism.

Proof. By construction the map T_u is $\mathbb{Q}_u$-linear with domain and codomain of the same dimension $dn = \sum_{v|u} d_v n$ as vector spaces over $\mathbb{Q}_u$. Moreover $T_u(\mathbb{Q}^{dn})$ is the image of K^n in $\prod_{v|u} K^n_v$ under the diagonal embedding, which is dense in this product. So T_u is surjective and therefore it is an isomorphism. This proves the first assertion, and the others follow from it. □

Proposition 11.2 (Thunder's principle)**.** *With the notation of Lemma 11.1, let $\mathcal{K}$ be a convex body of $K^n_{\mathbb{A}}$. Then, $\mathcal{C} := T^{-1}_{\mathbb{A}}(\mathcal{K})$ is a convex body of $\mathbb{Q}^{dn}_{\mathbb{A}}$ and we have*

$$\lambda_{d(i-1)+j}(\mathcal{C}) \asymp \lambda_i(\mathcal{K}) \quad (1 \le i \le n,\ 1 \le j \le d),$$

with implied constants that depend only on K, n and T.

Proof. Writing $\mathcal{K} = \prod_{v\in M(K)} \mathcal{K}_v$, we find that $\mathcal{C} = \prod_{u\in M(\mathbb{Q})} \mathcal{C}_u$ where

$$\mathcal{C}_u = \bigcap_{v|u} T^{-1}_v(\mathcal{K}_v)$$

is a convex body of $\mathbb{Q}^{dn}_u$ for each $u \in M(\mathbb{Q})$. For all but finitely many places $u \neq \infty$, we also have that $T_u(\mathbb{Z}^{dn}_u) = \prod_{v|u} \mathcal{O}^n_v = \prod_{v|u} \mathcal{K}_v$ and so $\mathcal{C}_u = \mathbb{Z}^{dn}_u$. Thus $\mathcal{C}$ is a convex body of $\mathbb{Q}^{dn}_{\mathbb{A}}$.

Choose linearly independent elements $\mathbf{y}_1, \dots, \mathbf{y}_n$ of K^n over K which realize the successive minima of $\mathcal{K}$, choose a basis $\boldsymbol{\omega} = (\omega_1, \dots, \omega_d)$ of the ring of integers $\mathcal{O}_K$ of K as a $\mathbb{Z}$-module, and let $c = \max\{|\boldsymbol{\omega}_j|_v\,;\, 1 \le j \le d \text{ and } v \mid \infty\}$. Then the points

$$\mathbf{x}_{i,j} := T^{-1}(\omega_j \mathbf{y}_i) \in \mathbb{Q}^{dn} \quad (1 \le i \le n,\ 1 \le j \le d)$$

are linearly independent over $\mathbb{Q}$. For each indexing pair (i,j), we also have $\mathbf{y}_i \in \lambda_i(\mathcal{K})\mathcal{K}$ and $\omega_j \mathcal{K} \subseteq c\mathcal{K}$, thus $\omega_j \mathbf{y}_i \in c\lambda_i(\mathcal{K})\mathcal{K}$. As T_∞ is linear over $\mathbb{Q}_\infty = \mathbb{R}$, this means that

$$\mathbf{x}_{i,j} \in T_{\mathbb{A}}^{-1}(c\lambda_i(\mathcal{K})\mathcal{K}) = c\lambda_i(\mathcal{K})\mathcal{C}.$$

In view of the linear independence of the points $\mathbf{x}_{i,j}$ and the fact that $\lambda_1(\mathcal{K}) \le \cdots \le \lambda_n(\mathcal{K})$, we conclude that

$$\lambda_{d(i-1)+j}(\mathcal{C}) \le c\lambda_i(\mathcal{K}) \quad (1 \le i \le n,\ 1 \le j \le d).$$

On the other hand, since $T_{\mathbb{A}}$ is $\mathbb{Q}_{\mathbb{A}}$-linear and invertible, we have $\mu(\mathcal{C}) = c'\mu(\mathcal{K})$ for some constant $c' > 0$ which depends only on $T_{\mathbb{A}}$. So, the adelic Minkowski's theorem 4.1, applied to $\mathcal{C}$ and $\mathcal{K}$ separately, yields

$$\lambda_1(\mathcal{C}) \cdots \lambda_{dn}(\mathcal{C}) \asymp \mu(\mathcal{C})^{-1} \asymp \mu(\mathcal{K})^{-1} \asymp (\lambda_1(\mathcal{K}) \cdots \lambda_n(\mathcal{K}))^d.$$

The conclusion follows. □

12. Proofs of Theorems B and C

We fix a basis $\boldsymbol{\alpha} = (\alpha_1, \dots, \alpha_d)$ of K over $\mathbb{Q}$, a place w of K of local degree $d_w = 1$ above a place ℓ of $\mathbb{Q}$, so that $K_w = \mathbb{Q}_\ell$, and a non-zero point $\boldsymbol{\xi} = (\xi_1, \dots, \xi_n) \in K_w^n = \mathbb{Q}_\ell^n$, assuming $n \ge 2$. We form

$$\Xi = \boldsymbol{\alpha} \otimes \boldsymbol{\xi} = \big(\alpha_1 \boldsymbol{\xi}, \dots, \alpha_d \boldsymbol{\xi}\big) \in (K_w^n)^d = (\mathbb{Q}_\ell^n)^d,$$

and note that

$$\| \Xi \|_\ell = \| \boldsymbol{\alpha} \|_w \| \boldsymbol{\xi} \|_w. \tag{12.1}$$

In order to apply the results of section 7, we need to adjust Definition 7.2 so that the family of convex bodies in $\mathbb{Q}_{\mathbb{A}}^{dn}$ attached to Ξ and the family of convex bodies in $K_{\mathbb{A}}^n$ attached to $\boldsymbol{\xi}$ do not depend on the norms of those points. Thus, for each $q \ge 0$, we define

$$\mathcal{C}_\Xi(q) = \left\{ (\mathbf{x}_u) \in \mathbb{Q}_{\mathbb{A}}^{dn}\ ;\ \| \Xi \|_\ell^{-1} |\mathbf{x}_\ell \cdot \Xi|_\ell \le e^{-q} \ \text{ and } \| \mathbf{x}_u \|_u \le 1 \text{ for each } u \in M(\mathbb{Q}) \right\},$$
$$\mathcal{C}_{\boldsymbol{\xi}}(q) = \left\{ (\mathbf{y}_v) \in K_{\mathbb{A}}^n\ ;\ \| \boldsymbol{\xi} \|_w^{-1} |\mathbf{y}_w \cdot \boldsymbol{\xi}|_w \le e^{-qd} \ \text{ and } \| \mathbf{y}_v \|_v \le 1 \text{ for each } v \in M(K) \right\},$$

since $d_w = 1$. In order to relate the minima of these convex bodies and to deduce relationships between the standard four exponents of approximation attached to the triples $(\Xi, \mathbb{Q}, \ell)$ and $(\boldsymbol{\xi}, K, w)$, we consider the $\mathbb{Q}$-linear isomorphism $T\colon (\mathbb{Q}^n)^d \to K^n$ given by

$$T(\mathbf{x}_1, \dots, \mathbf{x}_d) = \alpha_1 \mathbf{x}_1 + \cdots + \alpha_d \mathbf{x}_d$$

for any $\mathbf{x}_1, \dots, \mathbf{x}_d \in \mathbb{Q}^n$. For each place u of $\mathbb{Q}$ and each place v of K above u, it extends by continuity to a $\mathbb{Q}_u$-linear map T_v from $(\mathbb{Q}_u^n)^d$ to K_v^n given by

$$T_v(\mathbf{x}_1, \dots, \mathbf{x}_d) = \alpha_1 \mathbf{x}_1 + \cdots + \alpha_d \mathbf{x}_d$$

for any $\mathbf{x}_1, \dots, \mathbf{x}_d \in \mathbb{Q}_u^n$. Following Lemma 11.1, this yields a $\mathbb{Q}_u$-linear isomorphism T_u from $\mathbb{Q}_u^{nd}$ to $\prod_{v|u} K_v^n$, as well as a $\mathbb{Q}_{\mathbb{A}}$-linear isomorphism $T_{\mathbb{A}}$ from $\mathbb{Q}_{\mathbb{A}}^{nd}$ to $K_{\mathbb{A}}^n$.

Proposition 12.1. *There is an idèle $\boldsymbol{a} \in K_{\mathbb{A}}^*$ which depends only on K, w, $\boldsymbol{\alpha}$ and n such that, for each $q \geq 0$,*

$$\boldsymbol{a}^{-1}\mathcal{C}_{\Xi}(dq) \subseteq T_{\mathbb{A}}^{-1}(\mathcal{C}_{\boldsymbol{\xi}}(q)) \subseteq \boldsymbol{a}\,\mathcal{C}_{\Xi}(dq).$$

Proof. Fix a place u of $\mathbb{Q}$. Since T_u is a $\mathbb{Q}_u$-linear isomorphism, there exists a constant $c_u \geq 1$ such that

$$c_u^{-1}\|\,\mathbf{x}\,\|_u \leq \max_{v|u} \|\,T_v(\mathbf{x})\,\|_v \leq c_u\|\,\mathbf{x}\,\|_u$$

for all $\mathbf{x} \in \mathbb{Q}_u^{nd}$. Since $T_u(\mathbb{Z}_u^{nd}) = \prod_{v|u}\mathcal{O}_v^n$ for all but finitely many $u \neq \infty$, we may take $c_u = 1$ for those u. When $u = \ell$ and $\mathbf{x} = (\mathbf{x}_1, \dots, \mathbf{x}_d) \in (\mathbb{Q}_\ell^n)^d$, we also find

$$\mathbf{x}\cdot\Xi = \sum_{i=1}^{d} \alpha_i \mathbf{x}_i \cdot \boldsymbol{\xi} = T_w(\mathbf{x})\cdot\boldsymbol{\xi},$$

thus $|\mathbf{x}\cdot\Xi|_\ell = |T_w(\mathbf{x})\cdot\boldsymbol{\xi}|_w$. Then, using (12.1) and assuming that $c_\ell \geq \max\{\|\,\boldsymbol{\alpha}\,\|_w, \|\,\boldsymbol{\alpha}\,\|_w^{-1}\}$, we deduce that

$$c_\ell^{-1}\frac{|\mathbf{x}\cdot\Xi|_\ell}{\|\,\Xi\,\|_\ell} \leq \frac{|T_w(\mathbf{x})\cdot\boldsymbol{\xi}|_w}{\|\,\boldsymbol{\xi}\,\|_w} \leq c_\ell\frac{|\mathbf{x}\cdot\Xi|_\ell}{\|\,\Xi\,\|_\ell}.$$

Choose an idèle $\boldsymbol{a} = (a_u) \in \mathbb{Q}_{\mathbb{A}}^*$ such that $|a_u|_u \geq c_u$ for each $u \in M(\mathbb{Q})$. Then, if $\mathbf{x} = (\mathbf{x}_u) \in \mathbb{Q}_{\mathbb{A}}^n$ and $\mathbf{y} = (\mathbf{y}_v) \in K_{\mathbb{A}}^n$ are related by $\mathbf{y} = T_{\mathbb{A}}(\mathbf{x})$, the above estimates yield

$$\|\,a_u^{-1}\mathbf{x}_u\,\|_u \leq \max_{v|u}\|\,\mathbf{y}_v\,\|_v \leq \|\,a_u\mathbf{x}_u\,\|_u \quad\text{and}\quad \frac{|a_\ell^{-1}\mathbf{x}_\ell\cdot\Xi|_\ell}{\|\,\Xi\,\|_\ell} \leq \frac{|\mathbf{y}_w\cdot\boldsymbol{\xi}|_w}{\|\,\boldsymbol{\xi}\,\|_w} \leq \frac{|a_\ell\mathbf{x}_\ell\cdot\Xi|_\ell}{\|\,\Xi\,\|_\ell}.$$

So, if $\mathbf{y} \in \mathcal{C}_{\boldsymbol{\xi}}(q)$ (resp. $\boldsymbol{a}\mathbf{x} \in \mathcal{C}_{\Xi}(dq)$) for some $q \geq 0$, then $\boldsymbol{a}^{-1}\mathbf{x} \in \mathcal{C}_{\Xi}(dq)$ (resp. $\mathbf{y} \in \mathcal{C}_{\boldsymbol{\xi}}(q)$). This means that $\boldsymbol{a}^{-1}T_{\mathbb{A}}^{-1}(\mathcal{C}_{\boldsymbol{\xi}}(q)) \subseteq \mathcal{C}_{\Xi}(dq)$ (resp. $\boldsymbol{a}^{-1}\mathcal{C}_{\Xi}(dq) \subseteq T_{\mathbb{A}}^{-1}(\mathcal{C}_{\boldsymbol{\xi}}(q))$), as needed. □

By Proposition 5.3 and Thunder's principle (Proposition 11.2), the above result yields the following estimates.

Corollary 12.2. *For each $i = 1, \dots, n$, each $j = 1, \dots, d$ and each $q \geq 0$, we have*

$$\lambda_{d(i-1)+j}(\mathcal{C}_{\Xi}(dq)) \asymp \lambda_i(\mathcal{C}_{\boldsymbol{\xi}}(q)) \tag{12.2}$$

with implicit constants that depend only on K, $\boldsymbol{\alpha}$, n and w.

Proof of Theorem B. Fix i, j and q as above. Taking logarithms on both sides of (12.2) and using Lemma 7.4, we obtain that the absolute value of the difference

$$L_{\Xi,d(i-1)+j}(dq) - L_{\boldsymbol{\xi},i}(q) \tag{12.3}$$

is bounded above by a constant that depends only on K, $\boldsymbol{\alpha}$, n and w. Letting $i' = n+1-i$ and $j' = d+1-j$, Lemma 7.8 shows that the same applies to

$$L_{\Xi,d(i-1)+j}(dq) + L^*_{\Xi,d(i'-1)+j'}(dq) - dq \quad\text{and}\quad L_{\boldsymbol{\xi},i}(q) + L^*_{\boldsymbol{\xi},i'}(q) - q.$$

Subtracting from the first number the sum of the second and of (12.3), we obtain that

$$L^*_{\Xi,d(i'-1)+j'}(dq) - L^*_{\boldsymbol{\xi},i'}(q) - (d-1)q$$

also has absolute value bounded above by such a constant. As i' runs from 1 to n with i, and as j' runs from 1 to d with j, this proves the two inequalities of Theorem B. □

Corollary 12.3. *Upon writing $\widehat{\omega}(\boldsymbol{\xi})$ for $\widehat{\omega}(\boldsymbol{\xi}, K, w)$, $\widehat{\omega}(\Xi)$ for $\widehat{\omega}(\Xi, \mathbb{Q}, \ell)$, and similarly for the other exponents, we have*

$$d\big(\widehat{\omega}(\boldsymbol{\xi})+1\big)=\widehat{\omega}(\Xi)+1, \qquad d\big(\omega(\boldsymbol{\xi})+1\big)=\omega(\Xi)+1,$$
$$d\Big(\frac{1}{\widehat{\lambda}(\boldsymbol{\xi})}+1\Big)=\frac{1}{\widehat{\lambda}(\Xi)}+1, \qquad d\Big(\frac{1}{\lambda(\boldsymbol{\xi})}+1\Big)=\frac{1}{\lambda(\Xi)}+1.$$

Proof. By Theorem B, the ratios $L_{\Xi,1}(dq)/q$ and $L_{\boldsymbol{\xi},1}(q)/q$ have the same limit points as q goes to infinity. In particular, they have the same superior limit and the same inferior limit. Applying Lemma 10.1 separately to $(\Xi, \mathbb{Q}, \ell)$ and $(\boldsymbol{\xi}, K, w)$ to compute these limits and comparing the results, we get the first row of equalities.

By Theorem B, the quantities $L^*_{\Xi,1}(dq)/q - d$ and $L^*_{\boldsymbol{\xi},1}(q)/q - 1$ also have the same limit points as q goes to infinity and the same lemma yields the second row of equalities. □

As an application, suppose that $n = 3$ and that $\boldsymbol{\xi} \in K_w^3$ has linearly independent coordinates over K. Then, $\boldsymbol{\xi}$ satisfies Jarník's identity (1.3) by Corollary 10.3. Using the formulas of the above corollary, we deduce the identity (1.6) relating $\widehat{\lambda}(\Xi)$ and $\widehat{\omega}(\Xi)$.

***Proof of Theorem C*.** Let S denote the subset of K_w^3 from Theorem 2.2 of Bel. Since the supremum of $\widehat{\lambda}(\boldsymbol{\xi}, K, w)$ is $1/\gamma$ as $\boldsymbol{\xi}$ runs through S, the formulas of Corollary 12.3 imply that for the corresponding points $\Xi = \boldsymbol{\alpha} \otimes \boldsymbol{\xi}$, the supremum of $\widehat{\lambda}(\Xi, \mathbb{Q}, \ell)$ is $1/(d\gamma^2 - 1)$. Since the points $\boldsymbol{\xi}$ of S satisfy Jarník's identity (1.3), the supremum of $\widehat{\omega}(\boldsymbol{\xi}, K, w)$ is γ^2 as $\boldsymbol{\xi}$ runs through S. So, for the corresponding points $\Xi = \boldsymbol{\alpha} \otimes \boldsymbol{\xi}$, we find similarly that the supremum of $\widehat{\omega}(\Xi, \mathbb{Q}, \ell)$ is $d(\gamma^2 + 1) - 1$. This proves Theorem C because, for $\boldsymbol{\xi} = (1, \xi, \xi^2) \in S$, the $3d$ coordinates of $(\boldsymbol{\alpha}, \xi\boldsymbol{\alpha}, \xi^2\boldsymbol{\alpha})$ form a permutation of those of $\Xi = (\alpha_1\boldsymbol{\xi}, \dots, \alpha_d\boldsymbol{\xi})$ and so these two points have the same exponents of approximation. □

As a final remark, suppose that $\mathbf{P} = (P_1, \dots, P_n)$ is an n-system on $[0, \infty)$ for which $\mathbf{L}_{\boldsymbol{\xi}} - \mathbf{P}$ is bounded. Then the difference $\mathbf{L}_\Xi - \mathbf{R}$ is bounded for the function $\mathbf{R} = (R_1, \dots, R_{nd})$ from $[0, \infty)$ to $\mathbb{R}^{nd}$ given by

$$R_{d(i-1)+j}(q) = P_i(q/d) \quad (1 \le i \le n,\ 1 \le j \le d,\ q \ge 0).$$

If $d > 1$, this is not an nd-system because its components are piecewise linear with slopes 0 and $1/d$. However, it is a generalized nd-system in the sense of [18, Definition 4.5] and so it can easily be approximated uniformly by an nd-system as explained in [18, section 4].

References

[1] P. Bel, Approximation simultanée d'un nombre v-adique et de son carré par des nombres algébriques, *J. Number Theory* **133** (2013), 3362–3380.

[2] E. Bombieri and J. Vaaler, On Siegel's Lemma, *Invent. Math.* **73** (1983), 11–32.

[3] Y. Bugeaud, On simultaneous uniform approximation to a p-adic number and its square, *Proc. Amer. Math. Soc.* **138** (2010), 3821–3826.

[4] Y. Bugeaud, M. Laurent, On transfer inequalities in Diophantine approximation, II *Math. Z.* **265** (2010), 249–262.

[5] E. B. Burger, On Mahler's compound bodies, *J. Austral. Math. Soc. (series A)* **55** (1993), 183–215.

[6] H. Davenport and W. M. Schmidt, Approximation to real numbers by algebraic integers, *Acta Arith.* **15** (1969), 393–416.
[7] V. Jarník, Zum Khintchineschen Übertragungssatz, *Trudy Tbilisskogo mathematicheskogo instituta im. A. M. Razmadze = Travaux de l'Institut mathématique de Tbilissi* **3** (1938), 193–212.
[8] M. Laurent, Exponents of Diophantine approximation in dimension two, *Canad. J. Math.* **61** (2009), 165–189.
[9] M. Laurent, On transfer inequalities in Diophantine approximation, in: *Analytic Number Theory, Essays in honour of Klaus Roth*, Cambridge U. Press (2009), 306–314.
[10] K. Mahler, Inequalities for ideal bases in algebraic number fields, *J. Austral. Math. Soc.* **4** (1964), 425–448.
[11] A. Marnat, About Jarník's-type relation in higher dimension, *Ann. Inst. Fourier (Grenoble)* **68** (2018), 131–150.
[12] A. Marnat and N. G. Moshchevitin, An optimal bound for the ratio between ordinary and uniform exponents of Diophantine approximation, Mathematika **66** (2020), 818–854.
[13] R. B. McFeat, Geometry of numbers in adele spaces, *Dissertationes Math. Rozprawy Mat.* **88** (1971), 49 pp.
[14] D. Roy, Approximation simultanée d'un nombre et de son carré, *C. R. Acad. Sci., Paris, ser. I* **336** (2003), 1–6.
[15] D. Roy, Approximation to real numbers by cubic algebraic integers I, *Proc. London Math. Soc.* **88** (2004), 42–62.
[16] D. Roy, Rational approximation to real points on conics, *Ann. Inst. Fourier (Grenoble)* **63** (2013), 2331–2348.
[17] D. Roy, On Schmidt and Summerer parametric geometry of numbers, *Ann. of Math.* **182** (2015), 739–786.
[18] D. Roy, Spectrum of the exponents of best rational approximation, *Math. Z.* **283** (2016), 143–155.
[19] D. Roy and M. Waldschmidt, Parametric geometry of numbers in function fields, *Mathematika* **63** (2017), 1114–1135.
[20] W. M. Schmidt, On heights of algebraic subspaces and Diophantine approximations, *Ann. of Math.* **85** (1967), 430–472.
[21] W. M. Schmidt, *Diophantine approximation*, Lecture Notes in Math., vol. 785, Springer-Verlag, 1980.
[22] W. M. Schmidt and L. Summerer, Diophantine approximation and parametric geometry of numbers, *Monatsh. Math.* **169** (2013), 51–104.
[23] O. Teulié, Approximation d'un nombre p-adique par des nombres algébriques, *Acta Arith.* **102** (2002), 137–155.
[24] J. L. Thunder, Remarks on adelic geometry of numbers, in: *Number theory for the millennium, III*, (Urbana, IL, 2000), 253–259, A K Peters, 2002.
[25] D. Zelo, *Simultaneous approximation to real and p-adic numbers*, Ph.D. thesis, University of Ottawa, 2008, 145pp; http://hdl.handle.net/10393/29866.

Département de Mathématiques, Université d'Ottawa, 150 Louis Pasteur, Ottawa, Ontario K1N 6N5, Canada

Email address: anthony.poels@uottawa.ca

Département de Mathématiques, Université d'Ottawa, 150 Louis Pasteur, Ottawa, Ontario K1N 6N5, Canada

Email address: droy@uottawa.ca