RATIONAL APPROXIMATION TO REAL POINTS
ON QUADRATIC HYPERSURFACES

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ABSTRACT. Let Z be a quadratic hypersurface of $\mathbb{P}^n(\mathbb{R})$ defined over $\mathbb{Q}$ containing points whose coordinates are linearly independent over $\mathbb{Q}$. We show that, among these points, the largest exponent of uniform rational approximation is the inverse $1/\rho$ of an explicit Pisot number $\rho < 2$ depending only on n if the Witt index (over $\mathbb{Q}$) of the quadratic form q defining Z is at most 1, and that it is equal to 1 otherwise. Furthermore there are points of Z which realize this maximum. They constitute a countably infinite set in the first case, and an uncountable set in the second case. The proof for the upper bound $1/\rho$ uses a recent transference inequality of Marnat and Moshchevitin. In the case $n = 2$, we recover results of the second author while for $n > 2$, this completes recent work of Kleinbock and Moshchevitin.

1. INTRODUCTION

Nowadays, we have a good knowledge on how well points in projective n-space $\mathbb{P}^n(\mathbb{R})$ can be approximated by rational points i.e. by points of $\mathbb{P}^n(\mathbb{Q})$. Thanks to recent advances in parametric geometry of numbers and in metrical theory [2, 15, 16], we essentially know all possible ways in which a given point in $\mathbb{P}^n(\mathbb{R})$ behaves with respect to rational approximation and, in good cases, we also know the Hausdorff dimension of the set of exceptional points having a given pattern of approximation. However, the situation changes drastically if we restrict to points from a proper algebraic subset Z of $\mathbb{P}^n(\mathbb{R})$ defined over $\mathbb{Q}$. In particular, for the type of problem that we have in mind, very little is known about algebraic curves of $\mathbb{P}^n(\mathbb{R})$ defined over $\mathbb{Q}$ of degree $d \geq 2$ besides the case $d = 2$ treated in [14] which reduces to studying conics in $\mathbb{P}^2(\mathbb{R})$. In this paper, we extend the results of [14] to quadratic hypersurfaces of $\mathbb{P}^n(\mathbb{R})$ defined over $\mathbb{Q}$, thus completing recent work of Kleinbock and Moshchevitin [5]. We adopt here the projective setting as in [7, 14] because it is more conceptual and brings simplifications with respect to the traditional but equivalent affine point of view. The connection is explained in Section 5.

To each point ξ of $\mathbb{P}^n(\mathbb{R})$, one attaches two numbers which measure how well it is approximated by rational points. Following the convention of Bugeaud and Laurent in [1], they

are the exponent of *uniform* rational approximation $\hat{\lambda}(\xi)$ and the exponent of *best* rational approximation $\lambda(\xi)$. We recall their precise definition in the next section. In this study, we may assume that each representative of ξ in $\mathbb{R}^{n+1}$ has linearly independent coordinates over $\mathbb{Q}$. Then these exponents satisfy

$$\frac{1}{n} \le \hat{\lambda}(\xi) \le 1 \quad \text{and} \quad \hat{\lambda}(\xi) \le \lambda(\xi) \le \infty.$$

There is also a third inequality relating $\hat{\lambda}(\xi)$ and $\lambda(\xi)$. It was conjectured by Schmidt and Summerer at the end of Section 3 in [17], and was recently proved by Marnat and Moshchevitin in [8]. It plays a crucial role in [5] and here also but through a sharper form, in terms of measures of approximation, from joint work with Van Nguyen [10]. These results are recalled in Section 5.

For each algebraic subset Z of $\mathbb{P}^n(\mathbb{R})$, we denote by Z^{li} the set of points of Z whose representatives in $\mathbb{R}^{n+1}$ have $\mathbb{Q}$-linearly independent coordinates and, provided that this set is not empty, we are interested in the following important invariant

$$\hat{\lambda}(Z) := \sup\{\hat{\lambda}(\xi) \,;\, \xi \in Z^{\mathrm{li}}\} \in [1/n, 1].$$

A quadratic form on $\mathbb{Q}^{n+1}$ is a map $q \colon \mathbb{Q}^{n+1} \to \mathbb{Q}$ given by a homogeneous polynomial of $\mathbb{Q}[t_0, \dots, t_n]$ of degree 2. Its Witt index is the largest integer $m \ge 0$ such that $\mathbb{Q}^{n+1}$ contains an orthogonal sum of m hyperbolic planes for q (see Section 3). For each quadratic form q on $\mathbb{Q}^{n+1}$, we denote by $q_{\mathbb{R}} \colon \mathbb{R}^{n+1} \to \mathbb{R}$ its extension to $\mathbb{R}^{n+1}$ given by the same polynomial, and by $Z(q_{\mathbb{R}})$ the set of zeros of $q_{\mathbb{R}}$ in $\mathbb{P}^n(\mathbb{R})$. By a quadratic hypersurface of $\mathbb{P}^n(\mathbb{R})$ defined over $\mathbb{Q}$ we mean any non-empty proper subset of $\mathbb{P}^n(\mathbb{R})$ of this form. In terms of exponents of approximation, our main result reads as follows.

Theorem 1.1. *Let $n \ge 2$ be an integer, let Z be a quadratic hypersurface of $\mathbb{P}^n(\mathbb{R})$ defined over $\mathbb{Q}$, and let m be the Witt index of the quadratic form on $\mathbb{Q}^{n+1}$ defining Z. If Z^{li} is not empty, then*

$$\hat{\lambda}(Z) = \begin{cases} 1/\rho_n & \text{if } m \le 1, \\ 1 & \text{else,} \end{cases}$$

where $\rho_n \in (1, 2)$ denotes the unique positive root of the polynomial $x^n - (x^{n-1} + \cdots + x + 1)$. Moreover, the set $\{\xi \in Z^{\mathrm{li}} \,;\, \hat{\lambda}(\xi) = \hat{\lambda}(Z)\}$ is countably infinite if $m \le 1$ and uncountable otherwise.

In [5, Theorem 1a] (resp. [5, Theorem 2a]), Kleinbock and Moshchevitin prove the upper bound $\hat{\lambda}(Z) \le 1/\rho_n$ for all quadratic hypersurfaces Z of $\mathbb{P}^n(\mathbb{R})$ defined by quadratic forms q on $\mathbb{Q}^{n+1}$ of the type

$$(1.1) \quad q(t_0, t_1, \dots, t_n) = t_0^2 - f(t_1, \dots, t_n) \quad \left(\text{resp. } q(t_0, t_1, \dots, t_n) = t_0 t_n - f(t_1, \dots, t_{n-1})\right)$$

where f is a quadratic form on $\mathbb{Q}^n$ (resp. $\mathbb{Q}^{n-1}$) with no nontrivial zero. As we will see in Section 4, the general case of a non-degenerate quadratic form q of Witt index $m \le 1$ can

be reduced to these two special cases. Thus their results yield $\hat{\lambda}(Z) \leq 1/\rho_n$ in the non-degenerate case when $m \leq 1$. Theorem 1.1 shows that this extends with an equality to all quadratic forms of Witt index $m \leq 1$. For example, it applies to the degenerate quadratic forms $q(t_0, \ldots, t_N)$ given by the same formulas (1.1) for each integer $N > n$. In particular, for each $n \geq 2$, it yields $\hat{\lambda}(Z) = 1/\rho_n$ for the hypersurface Z of $\mathbb{P}^n(\mathbb{R})$ of equation $t_0^2 - 2t_1^2 = 0$ (resp. $t_0 t_2 - t_1^2 = 0$).

When $n = 2$, the number $\rho_n = \rho_2$ is the golden ratio and we automatically have $m \leq 1$. Then, the above theorem reduces to [14, Theorem 1.2]. In general, ρ_n is a Pisot number for each $n \geq 2$ (see Section 8).

In the next section we present a sharper version of the theorem dealing with measures of approximation to points of Z instead of the coarse estimation provided by exponents of approximation. In the case where $m \leq 1$, it leads to a notion of extremal points on Z generalizing the notion of extremal numbers from [12, 13].

2. Main result and notation

Fix an integer $n \geq 1$. In this paper we endow $\mathbb{R}^{n+1}$ with the standard structure of Euclidean space for which the canonical basis $(\mathbf{e}_0, \ldots, \mathbf{e}_n)$ is orthonormal. More generally, for each $k = 1, \ldots, n+1$, we endow its k-th exterior power $\bigwedge^k \mathbb{R}^{n+1}$ with the Euclidean space structure for which the products $\mathbf{e}_{i_1} \wedge \cdots \wedge \mathbf{e}_{i_k}$ with $0 \leq i_1 < \cdots < i_k \leq n$ form an orthonormal basis. In all cases we use the same symbol $\| \ \|$ to denote the associated norm.

We denote by $(x_0 : x_1 : \cdots : x_n)$ or simply by $[\mathbf{x}]$ the class in $\mathbb{P}^n(\mathbb{R})$ of a non-zero point $\mathbf{x} = (x_0, x_1, \ldots, x_n)$ of $\mathbb{R}^{n+1}$. Given non-zero points $\mathbf{x}, \mathbf{y} \in \mathbb{R}^{n+1}$, we define

$$\operatorname{dist}([\mathbf{x}], [\mathbf{y}]) := \operatorname{dist}(\mathbf{x}, \mathbf{y}) := \frac{\| \mathbf{x} \wedge \mathbf{y} \|}{\| \mathbf{x} \| \| \mathbf{y} \|}$$

and call this ratio the *projective distance* between $[\mathbf{x}]$ and $[\mathbf{y}]$ as it depends only on the classes of the points $\mathbf{x}$ and $\mathbf{y}$ in $\mathbb{P}^n(\mathbb{R})$. Geometrically, this is the sinus of the acute angle between the lines $\mathbb{R}\mathbf{x}$ and $\mathbb{R}\mathbf{y}$ spanned by $\mathbf{x}$ and $\mathbf{y}$ in $\mathbb{R}^{n+1}$. It is well-known that this yields a metric on $\mathbb{P}^n(\mathbb{R})$ as it satisfies the triangle inequality

$$\operatorname{dist}([\mathbf{x}], [\mathbf{z}]) \leq \operatorname{dist}([\mathbf{x}], [\mathbf{y}]) + \operatorname{dist}([\mathbf{y}], [\mathbf{z}])$$

for any $\mathbf{x}, \mathbf{y}, \mathbf{z} \in \mathbb{R}^{n+1} \setminus \{0\}$.

Let $\xi \in \mathbb{P}^n(\mathbb{R})$ and let $\boldsymbol{\xi} \in \mathbb{R}^{n+1}$ be a representative of ξ so that $\xi = [\boldsymbol{\xi}]$. For each non-zero $\mathbf{x} \in \mathbb{Z}^{n+1}$, we set

$$D_\xi(\mathbf{x}) := \frac{\| \mathbf{x} \wedge \boldsymbol{\xi} \|}{\| \boldsymbol{\xi} \|} = \| \mathbf{x} \| \operatorname{dist}(\xi, [\mathbf{x}])$$

and for each $X \geq 1$ we define

$$\mathcal{D}_\xi(X) := \min \left\{ D_\xi(\mathbf{x}) \, ; \, \mathbf{x} \in \mathbb{Z}^{n+1} \setminus \{0\} \text{ and } \| \mathbf{x} \| \leq X \right\}.$$

This minimum is achieved by a *primitive* point of $\mathbb{Z}^{n+1}$, that is a point whose coordinates are relatively prime as a set. Thus, upon defining the *height* of a point in $\mathbb{P}^n(\mathbb{Q})$ as the norm $\|\mathbf{x}\|$ of its primitive representatives $\pm\mathbf{x}$ in $\mathbb{Z}^{n+1}$, we view $\mathcal{D}_\xi(X)$ as a measure of approximation to ξ by rational points of height at most X. We define $\hat{\lambda}(\xi)$ (resp. $\lambda(\xi)$) to be the supremum of all $\lambda \in \mathbb{R}$ such that $\mathcal{D}_\xi(X) \leq X^{-\lambda}$ for each sufficiently large X (resp. for arbitrarily large values of X). Equivalently, $\hat{\lambda}(\xi)$ (resp. $\lambda(\xi)$) is the supremum of all $\lambda \in \mathbb{R}$ such that

$$\limsup_{X\to\infty} X^\lambda \mathcal{D}_\xi(X) < \infty \qquad \left(\text{resp. } \liminf_{X\to\infty} X^\lambda \mathcal{D}_\xi(X) < \infty\right).$$

With this notation, we can add the following precision to Theorem 1.1.

Theorem 2.1. *Under the hypotheses of Theorem 1.1, suppose that $m \leq 1$. Then we have*

  (i) $\limsup_{X\to\infty} X^{1/\rho_n}\mathcal{D}_\xi(X) > 0$ *for each* $\xi \in Z^{\mathrm{li}}$;

  (ii) $\limsup_{X\to\infty} X^{1/\rho_n}\mathcal{D}_\xi(X) < \infty$ *for infinitely many* $\xi \in Z^{\mathrm{li}}$.

Indeed, Part (i) implies that $\hat{\lambda}(\xi) \leq 1/\rho_n$ for each $\xi \in Z^{\mathrm{li}}$ while Part (ii) yields points $\xi \in Z^{\mathrm{li}}$ with $\hat{\lambda}(\xi) \geq 1/\rho_n$ and thus $\hat{\lambda}(\xi) = 1/\rho_n$. Altogether, this means that $\hat{\lambda}(Z) = 1/\rho_n$.

We say that the points $\xi \in Z^{\mathrm{li}}$ which satisfy the condition in Part (ii) are the *extremal points* of Z. For any such point, there exist constants $c_1 \geq c_2 > 0$ such that $\mathcal{D}_\xi(X) \leq c_1 X^{-1/\rho_n}$ for each sufficiently large X, as well as $\mathcal{D}_\xi(X) \geq c_2 X^{-1/\rho_n}$ for arbitrarily large values of X. This generalizes the notion of extremal numbers from [13] because ρ_2 is the golden ratio and thus the extremal points of the conic Z of $\mathbb{P}^2(\mathbb{R})$ defined by the quadratic form $t_0 t_2 - t_1^2$ of Witt index $m = 1$ are the points $(1 : \xi : \xi^2)$ where ξ is an extremal number.

For any subset E of $\mathbb{P}^n(\mathbb{Q})$ and any $X \geq 1$, we define

$$\mathcal{D}_\xi(X; E) := \min\left\{ D_\xi(\mathbf{x}) \,;\, \mathbf{x} \in \mathbb{Z}^{n+1} \setminus \{0\}, \ \|\mathbf{x}\| \leq X \text{ and } [\mathbf{x}] \in E \right\}$$

with the convention that $\min \emptyset = \infty$. Our main result below extends Theorem 2.1 by taking E to be either the set $Z(\mathbb{Q}) := Z \cap \mathbb{P}^n(\mathbb{Q})$ of rational points of Z or its complement.

Theorem 2.2. *Suppose that $n \geq 2$. Let Z be a quadratic hypersurface of $\mathbb{P}^n(\mathbb{R})$ defined over $\mathbb{Q}$ with $Z^{\mathrm{li}} \neq \emptyset$. Define $Z(\mathbb{Q}) := Z \cap \mathbb{P}^n(\mathbb{Q})$ and $E := \mathbb{P}^n(\mathbb{Q}) \setminus Z$. Then*

  (i) *we have* $\limsup_{X\to\infty} X^{1/\rho_n}\mathcal{D}_\xi(X; E) > 0$ *for each* $\xi \in Z^{\mathrm{li}}$;

  (ii) *we have* $\limsup_{X\to\infty} X^{1/\rho_n}\mathcal{D}_\xi(X; E) < \infty$ *for infinitely many* $\xi \in Z^{\mathrm{li}}$;

  (iii) *there exists $\epsilon > 0$ such that the set*

$$\{\xi \in Z^{\mathrm{li}} \,;\, \limsup_{X\to\infty} X^{1/\rho_n - \epsilon}\mathcal{D}_\xi(X; E) < \infty\}$$

   is at most countable.

Moreover, let m denote the Witt index of the quadratic form on $\mathbb{Q}^{n+1}$ defining Z.

(iv) *If $m \leq 1$, then any $\xi \in Z^{\mathrm{li}}$ with $\hat{\lambda}(\xi) > 1/2$ satisfies $\mathcal{D}_\xi(X) = \mathcal{D}_\xi(X; E)$ for each sufficiently large X.*

(v) *If $m > 1$, then any $\xi \in Z^{\mathrm{li}}$ with $\hat{\lambda}(\xi) > 1/\rho_n$ satisfies $\mathcal{D}_\xi(X) = \mathcal{D}_\xi(X; Z(\mathbb{Q}))$ for each sufficiently large X. Moreover, for each monotonically decreasing function $\varphi\colon [1, \infty) \to (0, 1]$ with $\lim_{X \to \infty} \varphi(X) = 0$ and $\lim_{X \to \infty} X\varphi(X) = \infty$, there are uncountably many $\xi \in Z^{\mathrm{li}}$ satisfying $\mathcal{D}_\xi(X; Z(\mathbb{Q})) \leq \varphi(X)$ for all sufficiently large X.*

In view of (iv), when $m \leq 1$, Parts (i) and (ii) yield Theorem 2.1 (because $1 < \rho_n < 2$). Moreover the extremal points of Z are the points $\xi \in Z^{\mathrm{li}}$ satisfying the inequality of Part (ii). Taking this condition as the general definition of an extremal point ξ of Z (without the above restriction $m \leq 1$), it follows from (iii) that the extremal points of Z always form an infinite countable set. Finally, if $m > 1$, then applying (v) with $\varphi = \log(3X)/X$ yields uncountably many points $\xi \in Z^{\mathrm{li}}$ with $\hat{\lambda}(\xi) \geq 1$ and so $\hat{\lambda}(\xi) = 1$. Thus Theorem 2.2 also implies Theorem 1.1.

The proof of Parts (i) and (iii) is given in Section 7 based on a result from joint work with Van Nguyen [10] that we recall in Section 5. This result complements the inequality of Marnat and Moshchevitin from [8], also recalled in Section 5, which relates $\lambda(\xi)$ and $\hat{\lambda}(\xi)$ for any $\xi \in (\mathbb{P}^n(\mathbb{R}))^{\mathrm{li}}$. The proof of Parts (ii), (iv) and (v) are given respectively in Sections 8, 9 and 10 respectively. The preliminary sections 3 and 4 present general facts about quadratic forms. In Section 3, we recall the Witt decomposition of a quadratic space V over $\mathbb{Q}$ and define a morphism $\psi\colon V \times V \to V$ with crucial properties. Section 4 reduces the study of a general quadratic form to two types of forms, a fact that we use in Section 8 for the construction of extremal points. Finally, Section 6 provides various estimates needed throughout the paper.

3. Preliminaries on quadratic forms

Let V be a vector space over $\mathbb{Q}$ of finite dimension $n + 1$ for some integer $n \geq 0$. A *quadratic form* on V is a map $q\colon V \to \mathbb{Q}$ given by

$$(3.1) \qquad q(\mathbf{x}) = \frac{1}{2} b(\mathbf{x}, \mathbf{x}) \quad \text{for each } \mathbf{x} \in V,$$

for some non-zero symmetric bilinear form $b\colon V \times V \to \mathbb{Q}$. This bilinear form is in turn uniquely determined by q through the formula

$$(3.2) \qquad b(\mathbf{x}, \mathbf{y}) = q(\mathbf{x} + \mathbf{y}) - q(\mathbf{x}) - q(\mathbf{y}) \quad \text{for each } \mathbf{x}, \mathbf{y} \in V.$$

When $V = \mathbb{Q}^{n+1}$, which is the main case of interest for us, this is equivalent to the definition given in the introduction. Moreover, the normalization factor $1/2$ in (3.1) ensures that, if q is integer-valued on $\mathbb{Z}^{n+1}$, then b also is integer-valued on $\mathbb{Z}^{n+1} \times \mathbb{Z}^{n+1}$ (by (3.2)).

A useful consequence of the formula (3.1) is that

$$(3.3) \qquad q(s\mathbf{x} + t\mathbf{y}) = s^2 q(\mathbf{x}) + stb(\mathbf{x}, \mathbf{y}) + t^2 q(\mathbf{y})$$

for any $s, t \in \mathbb{Q}$ and any $\mathbf{x}, \mathbf{y} \in V$. For the choice of $s = b(\mathbf{x}, \mathbf{y})$ and $t = -q(\mathbf{x})$, it yields the following generalization of [14, Lemma 4.1].

Lemma 3.1. *For each choice of* $\mathbf{x}, \mathbf{y} \in V$, *the point*

$$\mathbf{z} = \psi(\mathbf{x}, \mathbf{y}) := b(\mathbf{x}, \mathbf{y})\mathbf{x} - q(\mathbf{x})\mathbf{y} \in V$$

satisfies $q(\mathbf{z}) = q(\mathbf{x})^2 q(\mathbf{y})$ *and* $\psi(\mathbf{x}, \mathbf{z}) = q(\mathbf{x})^2 \mathbf{y}$.

In particular, we have $q(\mathbf{z}) = 0$ if $q(\mathbf{y}) = 0$ and $q(\mathbf{z}) = 1$ if $q(\mathbf{x}) = q(\mathbf{y}) = 1$. The polynomial map $\psi \colon V \times V \to V$ so defined is central to the present work. Note that it is bi-homogeneous of degree $(2, 1)$.

Given q and b as above, we say that points $\mathbf{x}$, $\mathbf{y}$ of V are *orthogonal* (with respect to q) if $b(\mathbf{x}, \mathbf{y}) = 0$. It is well-known that every subspace W of V admits an *orthogonal basis*, namely a basis whose elements are pairwise orthogonal. In particular, V itself admits an orthogonal basis $(\mathbf{x}_0, \ldots, \mathbf{x}_n)$ and upon setting $a_i = q(\mathbf{x}_i)$ for $i = 0, \ldots, n$, we obtain

$$(3.4) \qquad q(t_0 \mathbf{x}_0 + \cdots + t_n \mathbf{x}_n) = a_0 t_0^2 + \cdots + a_n t_n^2$$

for each $(t_0, \ldots, t_n) \in \mathbb{Q}^{n+1}$. A theorem of Silvester (valid more generally over $\mathbb{R}$) tells us that in such a formula the number n_0 (resp. n_+, resp. n_-) of indices i with $a_i = 0$ (resp. $a_i > 0$, resp. $a_i < 0$) is independent of the basis. The difference $n - n_0$ is called the *rank* of q.

The *orthogonal* of a subspace U of V is the subspace $U^\perp$ given by

$$U^\perp := \{\mathbf{x} \in V \, ; \, b(\mathbf{x}, \mathbf{y}) = 0 \text{ for each } \mathbf{y} \in U\}.$$

In particular the subspace $\ker(q) := V^\perp$ is called the *kernel* of q. It has dimension n_0. We say that q is *non-degenerate* if $\ker(q) = \{0\}$ and *degenerate* otherwise. In general, we say that a subspace U of V is *non-degenerate* if $U \cap U^\perp = \{0\}$, that it is *totally isotropic* if $U \subseteq U^\perp$, and that it is *anisotropic* if it contains no zero of q else than 0. A *hyperbolic plane* of V is any non-degenerate and non-anisotropic subspace of V of dimension 2. Equivalently, this is a subspace H of V which admits a basis $\{\mathbf{x}, \mathbf{y}\}$ with $q(\mathbf{x}) = q(\mathbf{y}) = 0$ and $b(\mathbf{x}, \mathbf{y}) \neq 0$. We say that subspaces U and W of V are *orthogonal* if $W \subseteq U^\perp$. Finally we say that a subspace W of V is an *orthogonal sum* of subspaces $W_1, \ldots, W_s$ if these subspaces are pairwise orthogonal and if W is their direct sum. We then express this property as $W = W_1 \perp \cdots \perp W_s$. A theorem of Witt [6, Chapter XIV, §5, Corollary 5] tells us that there is a unique integer $m \geq 0$, called the *Witt index* of q such that

$$V = \ker(q) \perp H_1 \perp \cdots \perp H_m \perp W$$

where $H_1, \ldots, H_m$ are hyperbolic planes of V and W is an anisotropic subspace of V. Another characterization of m is that all maximal totally isotropic subspaces of V have dimension $m + \dim_{\mathbb{Q}} \ker(q)$.

4. Equivalent forms

Two quadratic forms q and $\tilde{q}$ on $\mathbb{Q}^{n+1}$ are said to be *equivalent* if $\tilde{q} = q \circ T$ for some invertible linear operator T on $\mathbb{Q}^{n+1}$. Then q and $\tilde{q}$ have the same rank and the same Witt index. Moreover, the extended quadratic forms $q_{\mathbb{R}}$ and $\tilde{q}_{\mathbb{R}}$ on $\mathbb{R}^{n+1}$ satisfy $\tilde{q}_{\mathbb{R}} = q_{\mathbb{R}} \circ T_{\mathbb{R}}$ where $T_{\mathbb{R}}$ denotes the invertible linear operator on $\mathbb{R}^{n+1}$ extending T. For the associated zero sets $Z = Z(q_{\mathbb{R}})$ and $\tilde{Z} = Z(\tilde{q}_{\mathbb{R}})$ in $\mathbb{P}^n(\mathbb{R})$, this implies that

$$(4.1) \qquad Z = T_{\mathbb{R}}(\tilde{Z}), \quad Z^{\mathrm{li}} = T_{\mathbb{R}}(\tilde{Z}^{\mathrm{li}}) \quad \text{and} \quad Z \cap \mathbb{P}^n(\mathbb{Q}) = T(\tilde{Z} \cap \mathbb{P}^n(\mathbb{Q})),$$

using the same symbols T (resp. $T_{\mathbb{R}}$) to denote the automorphism of $\mathbb{P}^n(\mathbb{Q})$ (resp. $\mathbb{P}^n(\mathbb{R})$) induced by T (resp. $T_{\mathbb{R}}$).

Standard arguments as in [14, Section 2] also yield the following estimates.

Lemma 4.1. *Let $\xi \in \mathbb{P}^n(\mathbb{R})$ and let E be a non-empty subset of $\mathbb{P}^n(\mathbb{Q})$. Put $\tilde{\xi} = T_{\mathbb{R}}(\xi)$ and $\tilde{E} = T(E)$ for some $T \in \mathrm{GL}_{n+1}(\mathbb{Q})$. Then, for each $X \geq 1$, we have*

$$\mathcal{D}_{\tilde{\xi}}(X; \tilde{E}) \asymp \mathcal{D}_\xi(X; E)$$

with implied constants depending only on T.

Combining this lemma with the preceding observations, we deduce that, if Theorem 2.2 holds for some quadratic hypersurface Z attached to a quadratic form q on $\mathbb{Q}^{n+1}$, then it also holds for the quadratic hypersurface $\tilde{Z}$ attached to any quadratic form $\tilde{q}$ that is equivalent to q or more generally to aq for some $a \in \mathbb{Q}^\times$. This reduction will be useful in the proof of Theorem 2.2 (ii).

Keeping the same notation, we also deduce from (4.1) that Z (resp. Z^{li}) is not empty if and only if the same is true of $\tilde{Z}$ (resp. $\tilde{Z}^{\mathrm{li}}$). This observation yields the following criterion.

Proposition 4.2. *Let $q \colon \mathbb{Q}^{n+1} \to \mathbb{Q}$ be a (non-zero) quadratic form, and let $Z = Z(q_{\mathbb{R}})$. Then $Z^{\mathrm{li}} \neq \emptyset$ if and only if there exists $a \in \mathbb{Q}^\times$ such that aq is equivalent to*

$$(4.2) \qquad t_0^2 - a_1 t_1^2 - \cdots - a_n t_n^2$$

for some integers $a_1, \ldots, a_n$ where $a_1 > 0$ is not a square, or equivalent to

$$(4.3) \qquad t_0 t_1 - a_2 t_2^2 - \cdots - a_n t_n^2$$

for some integers $a_2, \ldots, a_n$ with $a_2 \neq 0$.

Note that the two cases are not mutually exclusive. However, in the second case, the Witt index of q is at least 1.

Proof. Suppose first that $Z^{\mathrm{li}} \neq \emptyset$. Then q is equivalent to $a_0 t_0^2 + \cdots + a_n t_n^2$ for some rational numbers $a_0, \ldots, a_n$ among which at least one is positive and at least one is negative. By permuting the variables if necessary we may assume that $a_0 > 0$ and $a_1 < 0$. If $-a_0 a_1$ is a square, the polynomial $a_0 t_0^2 + a_1 t_1^2$ is not irreducible over $\mathbb{Q}$ and then we may also assume that $a_2 \neq 0$. Hence by a further diagonal change of variables we obtain that $a_0 q$ is equivalent to $t_0^2 - a_1' t_1^2 - \cdots - a_n' t_n^2$ for some integers $a_1', \ldots, a_n'$ satisfying $a_1' > 0$ and also $a_2' \neq 0$ when a_1' is a square. In the latter case $t_0^2 - a_1' t_1^2$ is equivalent to $t_0 t_1$ and so $a_0 q$ is equivalent to $t_0 t_1 - a_2' t_2^2 - \cdots - a_n' t_n^2$ where $a_2' \neq 0$.

Conversely, assume that aq is equivalent to the quadratic form $\tilde{q}$ given by (4.2) or (4.3) for an appropriate choice of coefficients. Then $\tilde{q}_{\mathbb{R}}$ admits a zero $\boldsymbol{\xi} = (\xi_0, \xi_1, \ldots, \xi_n)$ in $\mathbb{R}^{n+1}$ with $\xi_1, \ldots, \xi_n$ algebraically independent over $\mathbb{Q}$. If the coordinates of $\boldsymbol{\xi}$ are not linearly independent over $\mathbb{Q}$, then $\xi_0 = c_1 \xi_1 + \cdots + c_n \xi_n$ for some $c_1, \ldots, c_n \in \mathbb{Q}$ and so $(\xi_1, \ldots, \xi_n)$ is a zero of the polynomial $\tilde{q}(c_1 t_1 + \cdots + c_n t_n, t_1, \ldots, t_n)$. However, this polynomial is non-zero because, if $\tilde{q}$ is given by (4.2), its coefficient of t_1^2 is $c_1^2 - a_1 \neq 0$ and, if $\tilde{q}$ is given by (4.3), its coefficient of t_2^2 is $-a_2 \neq 0$. This contradiction means that $[\boldsymbol{\xi}]$ belongs to $(Z(\tilde{q}_{\mathbb{R}}))^{\mathrm{li}}$ and so $Z^{\mathrm{li}} \neq \emptyset$. $\qquad \square$

The next corollary explains the assertion made in the introduction, to the effect that the results [5, Theorem 1a and 2a] of Kleinbock and Moshchevitin yield $\hat{\lambda}(Z) \leq 1/\rho_n$ for any quadratic hypersurface Z of $\mathbb{P}^n(\mathbb{R})$ with $Z^{\mathrm{li}} \neq \emptyset$, defined by a non-degenerate quadratic form on $\mathbb{Q}^{n+1}$ of Witt index $m \leq 1$. Note, on the way, that if $m = 0$ then we necessarily have $n \leq 3$ by a theorem of Meyer [18, Chapter IV, Section 3.2, Corollary 2].

Corollary 4.3. *Let $q \colon \mathbb{Q}^{n+1} \to \mathbb{Q}$ be a non-degenerate quadratic form, let m be its Witt index, and let $Z = Z(q_{\mathbb{R}})$. Then we have both $m \leq 1$ and $Z^{\mathrm{li}} \neq \emptyset$ if and only if there exists $a \in \mathbb{Q}^{\times}$ such that aq is equivalent to $t_0^2 - f(t_1, \ldots, t_n)$ for some quadratic form $f \colon \mathbb{Q}^n \to \mathbb{Q}$ with no non-trivial zero and $f(1, 0, \ldots, 0) > 0$, or equivalent to $t_0 t_1 - g(t_2, \ldots, t_n)$ for some quadratic form $g \colon \mathbb{Q}^{n-1} \to \mathbb{Q}$ with no non-trivial zero.*

Proof. This follows from the fact that a quadratic form on $\mathbb{Q}^{n+1}$ of the form (4.2) (resp. (4.3)) is non-degenerate of Witt index 0 or 1 if and only if the quadratic form $a_1 t_1^2 + \cdots + a_n t_n^2$ (resp. $a_2 t_2^2 + \cdots + a_n t_n^2$) has no non-trivial zero on $\mathbb{Q}^n$ (resp. $\mathbb{Q}^{n-1}$). $\qquad \square$

We conclude with the following result which derives from similar considerations.

Proposition 4.4. *Let $q \colon V \to \mathbb{Q}$ be a quadratic form of Witt index at least 1 and rank at least 3 on some finite dimensional vector space V over $\mathbb{Q}$. Then, for any finite set of proper subspaces of V, there is a zero of q in V which lies outside of their union.*

Proof. We have $\dim_{\mathbb{Q}}(V) = n + 1$ for some integer $n \geq 2$. By composing q with a suitable linear isomorphism from $\mathbb{Q}^{n+1}$ to V, we may assume that $V = \mathbb{Q}^{n+1}$ and that $q(t_0, \ldots, t_n) =$

$t_0 t_1 - a_2 t_2^2 - \cdots - a_n t_n^2$ for some $a_2, \ldots, a_n \in \mathbb{Q}$ with $a_2 \neq 0$. Then the polynomial map $\varphi \colon \mathbb{Q}^{n-1} \to \mathbb{Q}^{n+1}$ given by

$$\varphi(t_2, \ldots, t_n) = (1, a_2 t_2^2 + \cdots + a_n t_n^2, t_2, \ldots, t_n)$$

has image in the zero set of q in $\mathbb{Q}^{n+1}$. Moreover its components are linearly independent over $\mathbb{Q}$ as elements of $\mathbb{Q}[t_2, \ldots, t_n]$. Thus the composite $\ell \circ \varphi \colon \mathbb{Q}^{n+1} \to \mathbb{Q}$ is a non-zero polynomial function of degree at most 2 for each non-zero linear form ℓ on $\mathbb{Q}^{n+1}$. Consequently $\prod_{i=1}^{s} \ell_i \circ \varphi$ is a non-zero polynomial function of degree at most $2s$ on $\mathbb{Q}^{n+1}$ for any non-zero linear forms $\ell_1, \ldots, \ell_s$. This means that there is a point in the image of φ, thus a zero of q, which avoids $\bigcup_{i=1}^{s} \ker(\ell_i)$. The conclusion follows because any proper subspace of $\mathbb{Q}^{n+1}$ is contained in the kernel of a non-zero linear form on $\mathbb{Q}^{n+1}$. $\qquad\square$

Note that the above result also follows from the general avoidance lemma [4, Prop. 7.5] of Gaudron and Rémond, upon showing first that the set of zeros of q in V is not itself a union of hyperplanes of V.

5. A complement to the inequality of Marnat and Moshchevitin

We first explain how the definitions of Section 2 relate to those used in [10]. To this end, we fix a point $\boldsymbol{\xi} = (\xi_0, \ldots, \xi_n) \in \mathbb{R}^{n+1}$ with $\xi_0 \neq 0$, and we set $\xi = [\boldsymbol{\xi}]$. Then, for any $\mathbf{x} = (x_0, \ldots, x_n) \in \mathbb{Z}^{n+1}$, we have

(5.1)
$$D_\xi(\mathbf{x}) \asymp \max_{0 \le j < i \le n} |x_j \xi_i - x_i \xi_j| \asymp L_{\boldsymbol{\xi}}(\mathbf{x}) := \max_{1 \le i \le n} |x_0 \xi_i - x_i \xi_0|$$

where the implied constants depend only on $\boldsymbol{\xi}$ (and n). This yields

$$\mathcal{D}_\xi(X) \asymp \mathcal{L}_{\boldsymbol{\xi}}(X) := \min \left\{ L_{\boldsymbol{\xi}}(\mathbf{x}) \, ; \, \mathbf{x} \in \mathbb{Z}^{n+1} \setminus \{0\} \text{ and } \|\mathbf{x}\| \le X \right\} \quad (X \ge 1).$$

Thus we may replace $\mathcal{D}_\xi$ by $\mathcal{L}_{\boldsymbol{\xi}}$ in the definition of both $\lambda(\xi)$ and $\hat{\lambda}(\xi)$ as well as in the statement of Theorem 2.1. In particular, $\lambda(\xi)$ (resp. $\hat{\lambda}(\xi)$) is the supremum of all $\lambda \ge 0$ such that

$$\liminf_{X \to \infty} X^\lambda \mathcal{L}_{\boldsymbol{\xi}}(X) < \infty \quad (\text{resp. } \limsup_{X \to \infty} X^\lambda \mathcal{L}_{\boldsymbol{\xi}}(X) < \infty).$$

More generally, for any subset S of $\mathbb{Z}^{n+1}$ and any $X \ge 1$, we define

$$\mathcal{L}_{\boldsymbol{\xi}}(X; S) := \min \left\{ L_{\boldsymbol{\xi}}(\mathbf{x}) \, ; \, \mathbf{x} \in S \setminus \{0\} \text{ and } \|\mathbf{x}\| \le X \right\}$$

(with the convention that $\min \emptyset = \infty$). Then, for any non-empty subset E of $\mathbb{P}^n(\mathbb{Q})$, we have

$$\mathcal{D}_\xi(X; E) \asymp \mathcal{L}_{\boldsymbol{\xi}}(X; S),$$

where $S = \{\mathbf{x} \in \mathbb{Z}^{n+1} \setminus \{0\} \, ; \, [\mathbf{x}] \in E\}$.

We now quote the following general result from joint work with Van Nguyen [10]; it will be our main tool in proving Parts (i) and (iii) of Theorem 2.2.

Theorem 5.1. *Let $\boldsymbol{\xi}$ be a point of $\mathbb{R}^{n+1}$ whose coordinates are linearly independent over $\mathbb{Q}$ and let $S \subseteq \mathbb{Z}^{n+1}$. Suppose that $n \geq 2$ and that there exist positive real numbers a, b, α, β such that*

$$(5.2) \qquad bX^{-\beta} \leq \mathcal{L}_{\boldsymbol{\xi}}(X; S) \leq aX^{-\alpha}$$

for each sufficiently large real number X. Then we have $\alpha \leq \beta$ and

$$(5.3) \qquad \epsilon := 1 - \left(\alpha + \frac{\alpha^2}{\beta} + \cdots + \frac{\alpha^n}{\beta^{n-1}} \right) \geq 0.$$

Moreover, there exists a constant $C > 0$ which depends only on $\boldsymbol{\xi}, a, b, \alpha, \beta$ with the following property. If

$$(5.4) \qquad \epsilon \leq \frac{1}{4n} \left(\frac{\alpha}{\beta} \right)^n \min\{\alpha, \beta - \alpha\},$$

then there is an unbounded sequence $(\mathbf{y}_i)_{i \geq 0}$ of non-zero integer points in S which for each $i \geq 0$ satisfies the following conditions:

(i) $\left| \alpha \log \| \mathbf{y}_{i+1} \| - \beta \log \| \mathbf{y}_i \| \right| \leq C + 4\epsilon (\beta/\alpha)^n \log \| \mathbf{y}_{i+1} \|$;

(ii) $\left| \log L_{\boldsymbol{\xi}}(\mathbf{y}_i) + \beta \log \| \mathbf{y}_i \| \right| \leq C + 4\epsilon (\beta/\alpha)^2 \log \| \mathbf{y}_i \|$;

(iii) $\det(\mathbf{y}_i, \ldots, \mathbf{y}_{i+n}) \neq 0$;

(iv) *there exists no $\mathbf{x} \in S \cap \mathbb{Z}^{n+1} \setminus \{0\}$ with $\| \mathbf{x} \| < \| \mathbf{y}_i \|$ and $L_{\boldsymbol{\xi}}(\mathbf{x}) \leq L_{\boldsymbol{\xi}}(\mathbf{y}_i)$.*

Let $\xi = [\boldsymbol{\xi}]$. Then, as explained in [10], the first assertion of the theorem yields the inequality

$$\hat{\lambda}(\xi) + \frac{\hat{\lambda}(\xi)^2}{\lambda(\xi)} + \cdots + \frac{\hat{\lambda}(\xi)^n}{\lambda(\xi)^{n-1}} \leq 1$$

due to Marnat and Moshchevitin [8, Theorem 1], where the ratio $\hat{\lambda}(\xi)/\lambda(\xi)$ is interpreted as 0 when $\lambda(\xi) = \infty$. Note that these authors work in an affine setting which amounts to write $\xi = (1 : \xi_1 : \cdots : \xi_n)$ with $\xi_0 = 1$.

Corollary 5.2. *With the same notation, suppose further that $\epsilon = 0$. Then we have*

$$\limsup_{X \to \infty} X^\alpha \mathcal{L}_{\boldsymbol{\xi}}(X; S) > 0.$$

This follows from [10, Theorem 1.2] but it is instructive to derive it directly from the above theorem since similar arguments will be needed later. The proof given below uses standard estimates for a determinant as in [3, Lemma 9] (see Section 6).

Proof. If $\alpha = \beta$, the conclusion is immediate since (5.2) then yields $X^\alpha \mathcal{L}_{\boldsymbol{\xi}}(X; S) \geq b$ for each large enough X. We may therefore assume that $\alpha < \beta$. Since $\epsilon = 0$, the sequence $(\mathbf{y}_i)_{i \geq 0}$ provided by Theorem 5.1 satisfies

$$\| \mathbf{y}_{i+1} \| \asymp \| \mathbf{y}_i \|^{\beta/\alpha} \quad \text{and} \quad L_{\boldsymbol{\xi}}(\mathbf{y}_i) \asymp \| \mathbf{y}_i \|^{-\beta}$$

with implied constants that are independent of i. Put $X = \|\mathbf{y}_{i+n}\|/2$ for some arbitrarily large index i and choose a non-zero point $\mathbf{x}$ in S with $\|\mathbf{x}\| \leq X$, such that $L_{\boldsymbol{\xi}}(\mathbf{x}) = \mathcal{L}_{\boldsymbol{\xi}}(X; S)$. Assuming i large enough, the points $\mathbf{y}_i, \dots, \mathbf{y}_{i+n}$ are linearly independent over $\mathbb{Q}$ (by (iii)) with

$$\|\mathbf{y}_i\| < \|\mathbf{y}_{i+1}\| < \cdots < \|\mathbf{y}_{i+n}\| \quad \text{and} \quad L_{\boldsymbol{\xi}}(\mathbf{y}_i) > L_{\boldsymbol{\xi}}(\mathbf{y}_{i+1}) > \cdots > L_{\boldsymbol{\xi}}(\mathbf{y}_{i+n})$$

(since $\beta/\alpha > 1$). Moreover, since $\|\mathbf{x}\| < \|\mathbf{y}_{i+n}\|$, we have $L_{\boldsymbol{\xi}}(\mathbf{x}) > L_{\boldsymbol{\xi}}(\mathbf{y}_{i+n})$ by the minimality condition (iv). In particular, the points $\mathbf{y}_{i+n}$ and $\mathbf{x}$ are linearly independent and so there exists an index j with $i \leq j < i + n$ such that $\mathbf{y}_i, \dots, \widehat{\mathbf{y}_j}, \dots, \mathbf{y}_{i+n}, \mathbf{x}$ form a basis of $\mathbb{Q}^{n+1}$, where the hat on $\mathbf{y}_j$ means that this point is omitted from the list. As $\mathbf{y}_{i+n}$ realizes the maximum norm and the smallest value for $L_{\boldsymbol{\xi}}$ among these integer points, we deduce that

$$\begin{aligned}
1 \leq |\det(\mathbf{y}_i, \dots, \widehat{\mathbf{y}_j}, \dots, \mathbf{y}_{i+n}, \mathbf{x})| &\ll \|\mathbf{y}_{i+n}\| L_{\boldsymbol{\xi}}(\mathbf{y}_i) \cdots \widehat{L_{\boldsymbol{\xi}}(\mathbf{y}_j)} \cdots L_{\boldsymbol{\xi}}(\mathbf{y}_{i+n-1}) L_{\boldsymbol{\xi}}(\mathbf{x}) \\
&\ll \|\mathbf{y}_{i+n}\| L_{\boldsymbol{\xi}}(\mathbf{y}_i) \cdots L_{\boldsymbol{\xi}}(\mathbf{y}_{i+n-2}) L_{\boldsymbol{\xi}}(\mathbf{x}) \\
&\ll X^{1-\beta(\alpha^n/\beta^n + \cdots + \alpha^2/\beta^2)} \mathcal{L}_{\boldsymbol{\xi}}(X; S) = X^{\alpha} \mathcal{L}_{\boldsymbol{\xi}}(X; S).
\end{aligned}$$

The conclusion follows by letting i tend to infinity. $\qquad\square$

6. Metrical estimates

Let $\boldsymbol{\xi} = (\xi_0, \dots, \xi_n) \in \mathbb{R}^{n+1}$ with $\xi_0 \neq 0$, and let q be a quadratic form on $\mathbb{Q}^{n+1}$ with associated symmetric bilinear form b. In this section, we collect several estimates for polynomial maps which follow from Taylor expansions about the point $\boldsymbol{\xi}$. We start with the following generalization of [14, Lemma 4.2].

Lemma 6.1. *Suppose that $q(\boldsymbol{\xi}) = 0$. Then, for each $\mathbf{x}, \mathbf{y} \in \mathbb{Z}^{n+1}$, we have*

(i) $|b(\mathbf{x}, \mathbf{y})| \ll \|\mathbf{y}\| L_{\boldsymbol{\xi}}(\mathbf{x}) + \|\mathbf{x}\| L_{\boldsymbol{\xi}}(\mathbf{y})$,

(ii) $|q(\mathbf{x})| \ll \|\mathbf{x}\| L_{\boldsymbol{\xi}}(\mathbf{x})$,

and the point $\mathbf{z} := \psi(\mathbf{x}, \mathbf{y}) = b(\mathbf{x}, \mathbf{y})\mathbf{x} - q(\mathbf{x})\mathbf{y}$ satisfies

(iii) $L_{\boldsymbol{\xi}}(\mathbf{z}) \ll \|\mathbf{y}\| L_{\boldsymbol{\xi}}(\mathbf{x})^2 + \|\mathbf{x}\| L_{\boldsymbol{\xi}}(\mathbf{x}) L_{\boldsymbol{\xi}}(\mathbf{y})$,

(iv) $\|\mathbf{z}\| \ll \|\mathbf{x}\|^2 L_{\boldsymbol{\xi}}(\mathbf{y}) + \|\mathbf{y}\| L_{\boldsymbol{\xi}}(\mathbf{x})^2 + \|\mathbf{x}\| L_{\boldsymbol{\xi}}(\mathbf{x}) L_{\boldsymbol{\xi}}(\mathbf{y})$,

all implied constants depending only on q and $\boldsymbol{\xi}$.

Proof. We may assume that $\xi_0 = 1$. Then, upon denoting by x_0 the first coordinate of $\mathbf{x}$, we have $L_{\boldsymbol{\xi}}(\mathbf{x}) \asymp \|\Delta\mathbf{x}\|$ where $\Delta\mathbf{x} = \mathbf{x} - x_0\boldsymbol{\xi}$. Using similar notation for $\mathbf{y}$ and $\mathbf{z}$, we find

$$(6.1) \qquad b(\mathbf{x}, \mathbf{y}) = b(x_0\boldsymbol{\xi} + \Delta\mathbf{x}, y_0\boldsymbol{\xi} + \Delta\mathbf{y}) = y_0 b(\Delta\mathbf{x}, \boldsymbol{\xi}) + x_0 b(\boldsymbol{\xi}, \Delta\mathbf{y}) + b(\Delta\mathbf{x}, \Delta\mathbf{y})$$

which yields (i). As a special case, we obtain

$$(6.2) \qquad q(\mathbf{x}) = \frac{1}{2}b(\mathbf{x}, \mathbf{x}) = x_0 b(\Delta \mathbf{x}, \boldsymbol{\xi}) + q(\Delta \mathbf{x})$$

which in turn yields (ii). Since $\mathbf{z} = b(\mathbf{x}, \mathbf{y})\mathbf{x} - q(\mathbf{x})\mathbf{y}$, we get

$$L_{\boldsymbol{\xi}}(\mathbf{z}) \asymp \|\Delta \mathbf{z}\| = \|b(\mathbf{x}, \mathbf{y})\Delta \mathbf{x} - q(\mathbf{x})\Delta \mathbf{y}\| \ll |b(\mathbf{x}, \mathbf{y})|L_{\boldsymbol{\xi}}(\mathbf{x}) + |q(\mathbf{x})|L_{\boldsymbol{\xi}}(\mathbf{y})$$

which, together with the estimates (i) and (ii), leads to (iii). Finally, using (6.1) and (6.2), we obtain

$$z_0 = b(\mathbf{x}, \mathbf{y})x_0 - q(\mathbf{x})y_0 = x_0^2 b(\boldsymbol{\xi}, \Delta \mathbf{y}) + x_0 b(\Delta \mathbf{x}, \Delta \mathbf{y}) - y_0 q(\Delta \mathbf{x})$$

which implies $|z_0| \ll \|\mathbf{x}\|^2 L_{\boldsymbol{\xi}}(\mathbf{y}) + \|\mathbf{y}\|L_{\boldsymbol{\xi}}(\mathbf{x})^2$ and (iv) follows since $\|\mathbf{z}\| \leq \|z_0\boldsymbol{\xi}\| + \|\Delta \mathbf{z}\|$. $\qquad \square$

Similarly, using the multilinearity of the wedge product, we obtain

$$(6.3) \qquad \|\mathbf{x}_1 \wedge \cdots \wedge \mathbf{x}_k\| \ll \sum_{i=1}^{k} \|\mathbf{x}_i\| L_{\boldsymbol{\xi}}(\mathbf{x}_1) \cdots \widehat{L_{\boldsymbol{\xi}}(\mathbf{x}_i)} \cdots L_{\boldsymbol{\xi}}(\mathbf{x}_k)$$

for any $k = 1, \ldots, n+1$ and any $\mathbf{x}_1, \ldots, \mathbf{x}_k \in \mathbb{Z}^{n+1}$, with implied constants depending only on $\boldsymbol{\xi}$ and n. When $\|\mathbf{x}_k\| = \max_i \|\mathbf{x}_i\|$ and $L_{\boldsymbol{\xi}}(\mathbf{x}_k) = \min_i L_{\boldsymbol{\xi}}(\mathbf{x}_i)$, this simplifies to

$$\|\mathbf{x}_1 \wedge \cdots \wedge \mathbf{x}_k\| \ll \|\mathbf{x}_k\| L_{\boldsymbol{\xi}}(\mathbf{x}_1) \cdots L_{\boldsymbol{\xi}}(\mathbf{x}_{k-1}).$$

For $k = n+1$, these represent upper bounds for $|\det(\mathbf{x}_1, \ldots, \mathbf{x}_{n+1})|$ as in [3, Lemma 9].

We will also need the inequality

$$(6.4) \qquad \|(\mathbf{u} \cdot \mathbf{x})\mathbf{y} - (\mathbf{u} \cdot \mathbf{y})\mathbf{x}\| \leq \|\mathbf{u}\|\|\mathbf{x} \wedge \mathbf{y}\|$$

valid for any $\mathbf{u}, \mathbf{x}, \mathbf{y} \in \mathbb{R}^{n+1}$, where the dot represents the usual scalar product in $\mathbb{R}^{n+1}$. This is immediate when $\mathbf{u}$ is the point $\mathbf{e}_1 = (1, 0, \ldots, 0)$, and the general case follows by applying to all vectors a rotation mapping $\mathbf{u}$ to $\|\mathbf{u}\|\mathbf{e}_1$.

We conclude with the following simple criterion of linear independence.

Lemma 6.2. *Suppose that $\lim_{i \to \infty} D_\xi(\mathbf{x}_i) = 0$ for a point ξ in $\mathbb{P}^n(\mathbb{R})$ and a sequence of non-zero integer points $(\mathbf{x}_i)_{i \geq 1}$ in $\mathbb{Z}^{n+1}$. Then, the representatives $\boldsymbol{\xi}$ of ξ in $\mathbb{R}^{n+1}$ have linearly independent coordinates over $\mathbb{Q}$ if and only if the subsequence $(\mathbf{x}_i)_{i \geq i_0}$ spans $\mathbb{Q}^{n+1}$ for each $i_0 \geq 1$.*

Proof. There is no loss of generality in choosing $\boldsymbol{\xi}$ with $\|\boldsymbol{\xi}\| = 1$. Then, for each $\mathbf{u} \in \mathbb{Z}^{n+1}$, the inequality (6.4) yields

$$(6.5) \qquad \|(\mathbf{u} \cdot \mathbf{x}_i)\boldsymbol{\xi} - (\mathbf{u} \cdot \boldsymbol{\xi})\mathbf{x}_i\| \leq \|\mathbf{u}\|\|\mathbf{x}_i \wedge \boldsymbol{\xi}\| = \|\mathbf{u}\|D_\xi(\mathbf{x}_i) \quad (i \geq 1).$$

If $\boldsymbol{\xi}$ has linearly dependent coordinates, we may choose $\mathbf{u} \neq 0$ with $\mathbf{u} \cdot \boldsymbol{\xi} = 0$. Then (6.5) simplifies to $|\mathbf{u} \cdot \mathbf{x}_i| \leq \|\mathbf{u}\|D_\xi(\mathbf{x}_i)$. So the integer $\mathbf{u} \cdot \mathbf{x}_i$ vanishes for all sufficiently large i, say for each $i \geq i_0$, and therefore $(\mathbf{x}_i)_{i \geq i_0}$ does not span $\mathbb{Q}^{n+1}$. Conversely, if the subsequence

$(\mathbf{x}_i)_{i \geq i_0}$ does not span $\mathbb{Q}^{n+1}$ for some $i_0 \geq 1$, we may choose $\mathbf{u} \neq 0$ such that $\mathbf{u} \cdot \mathbf{x}_i = 0$ for each $i \geq i_0$. Then (6.5) yields $|\mathbf{u} \cdot \boldsymbol{\xi}| \leq |\mathbf{u} \cdot \boldsymbol{\xi}| \, \|\mathbf{x}_i\| \leq \|\mathbf{u}\| D_\xi(\mathbf{x}_i)$ for each $i \geq i_0$. This implies that $\mathbf{u} \cdot \boldsymbol{\xi} = 0$, and so $\boldsymbol{\xi}$ has linearly dependent coordinates. $\qquad\square$

7. Approximation by rational points outside of Z

The goal of this section is to prove Parts (i) and (iii) of Theorem 2.2. So, we assume that $n \geq 2$ and we fix a quadratic form q on $\mathbb{Q}^{n+1}$. We denote by m the Witt index of q and by $Z = Z(q_{\mathbb{R}}) \subseteq \mathbb{P}^n(\mathbb{R})$ the corresponding quadratic hypersurface. We also assume that Z^{li} is not empty, and define

$$E = \mathbb{P}^n(\mathbb{Q}) \setminus Z \quad \text{and} \quad S = \{\mathbf{x} \in \mathbb{Z}^{n+1} \, ; \, q(\mathbf{x}) \neq 0\} = \{\mathbf{x} \in \mathbb{Z}^{n+1} \setminus \{0\} \, ; \, [\mathbf{x}] \in E\}.$$

Moreover, we assume that q is integer valued on $\mathbb{Z}^{n+1}$ upon multiplying it by a suitable positive integer if necessary (this does not affect the hypersurface Z nor the Witt index m). Finally, we set $\rho = \rho_n$ (as defined in Theorem 1.1). We start with the following simple but crucial observation.

Lemma 7.1. *Let $\boldsymbol{\xi} \in \mathbb{R}^{n+1}$ be a representative of a point $\xi \in Z$. Then, there is a constant $b > 0$ such that $\mathcal{L}_{\boldsymbol{\xi}}(X; S) \geq bX^{-1}$ for each sufficiently large X.*

Proof. For each $\mathbf{x} \in S$, we have $|q(\mathbf{x})| \geq 1$ since $q(\mathbf{x})$ is a non-zero integer. By Lemma 6.1, we also have $|q(\mathbf{x})| \leq b^{-1} \|\mathbf{x}\| \mathcal{L}_{\boldsymbol{\xi}}(\mathbf{x})$ for a constant $b > 0$ depending only on $\boldsymbol{\xi}$ and q. Thus $L_{\boldsymbol{\xi}}(\mathbf{x}) \geq b \|\mathbf{x}\|^{-1}$ for each $\mathbf{x} \in S$ and so $\mathcal{L}_{\boldsymbol{\xi}}(X; S) \geq bX^{-1}$ for each $X \geq 1$. $\qquad\square$

We can now prove Theorem 2.2 (i).

Proposition 7.2. *For each $\xi \in Z^{\mathrm{li}}$, we have $\limsup_{X \to \infty} X^{1/\rho} \mathcal{D}_\xi(X; E) > 0$.*

Proof. Let $\boldsymbol{\xi} \in \mathbb{R}^{n+1}$ be a representative of ξ. If $\mathcal{L}_{\boldsymbol{\xi}}(X; S) > X^{-1/\rho}$ for arbitrarily large values of X, then $\limsup X^{1/\rho} \mathcal{L}_{\boldsymbol{\xi}}(X; S) > 0$ and we are done since $\mathcal{D}_\xi(X; E) \asymp \mathcal{L}_{\boldsymbol{\xi}}(X; S)$ (see Section 6). Thus, we may assume that $\mathcal{L}_{\boldsymbol{\xi}}(X; S) \leq X^{-1/\rho}$ for each sufficiently large X. Then, by the above lemma, Condition (5.2) of Theorem 5.1 holds with $\alpha = 1/\rho$, $\beta = 1$, $a = 1$ and some $b > 0$. By definition of $\rho = \rho_n$, the corresponding value for ϵ is $1 - (\alpha + \alpha^2 + \cdots + \alpha^n) = 0$. Thus Corollary 5.2 yields once again that $\limsup X^{1/\rho} \mathcal{L}_{\boldsymbol{\xi}}(X; S) > 0$. $\qquad\square$

Applying Theorem 5.1, we also deduce the following statement.

Proposition 7.3. *For each sufficiently small $\eta > 0$, there exists $\delta > 0$ with the following property. If a point $\xi \in Z^{\mathrm{li}}$ satisfies*

$$(7.1) \qquad \limsup_{X \to \infty} X^{1/\rho - \delta} \mathcal{D}_\xi(X; E) < \infty$$

then there is an unbounded sequence $(\mathbf{y}_i)_{i \geq 0}$ of primitive integer points in S which for each sufficiently large index i satisfy

(i) $\|\mathbf{y}_i\|^{\rho-\eta} \leq \|\mathbf{y}_{i+1}\| \leq \|\mathbf{y}_i\|^{\rho+\eta}$;

(ii) $D_\xi(\mathbf{y}_i) \leq \|\mathbf{y}_i\|^{-1+\eta}$;

(iii) $\det(\mathbf{y}_i, \ldots, \mathbf{y}_{i+n}) \neq 0$;

(iv) $\mathbf{y}_{i+1}$ is a rational multiple of $\psi(\mathbf{y}_i, \mathbf{y}_{i-n})$.

Assuming that $\eta \in (0, 2)$, Condition (ii) implies that the sequence of points $[\mathbf{y}_i]$ in $\mathbb{P}^n(\mathbb{Q})$ converges to ξ in $\mathbb{P}^n(\mathbb{R})$ because

$$\mathrm{dist}(\xi, [\mathbf{y}_i]) = \|\mathbf{y}_i\|^{-1} D_\xi(\mathbf{y}_i) \leq \|\mathbf{y}_i\|^{-2+\eta}.$$

Moreover Condition (iv) shows that this sequence is uniquely determined by its first terms since ψ is a bi-homogeneous map. As there are countably finite sequences in $\mathbb{P}^n(\mathbb{Q})$, we conclude that there are at most countably many points $\xi \in Z^{\mathrm{li}}$ which satisfy (7.1) for the corresponding δ and this proves Theorem 2.2 (iii).

Proof. Choose an arbitrarily small $\delta \in (0, 1/2)$ and assume that a point $\xi \in Z^{\mathrm{li}}$ with representative $\boldsymbol{\xi} \in \mathbb{R}^{n+1}$ satisfies (7.1). Then, this assumption together with Lemma 7.1 implies that Condition (5.2) of Theorem 5.1 holds with $\alpha = 1/\rho - \delta$, $\beta = 1$ and some $a, b > 0$. Since the corresponding $\epsilon = 1 - (\alpha + \cdots + \alpha^n)$ vanishes for $\alpha = 1/\rho$, Condition (5.4) is also fulfilled if δ is small enough as a function of n alone. Then, the theorem provides an unbounded sequence $(\mathbf{y}_i)_{i\geq 0}$ of integer points in S which satisfy the above conditions (i) to (iii) with $\eta = \mathcal{O}_n(\delta)$ for each sufficiently large i, say for each $i \geq i_0$. Moreover these points $\mathbf{y}_i$ are primitive because of their minimality property stated in Theorem 5.1 (iv).

Finally, let $\mathbf{z} = \psi(\mathbf{y}_i, \mathbf{y}_{i+1})$ for some $i \geq 2n + i_0$. Using Lemma 6.1, we find that

$$L_{\boldsymbol{\xi}}(\mathbf{z}) \ll \|\mathbf{y}_{i+1}\| L_{\boldsymbol{\xi}}(\mathbf{y}_i)^2 + \|\mathbf{y}_i\| L_{\boldsymbol{\xi}}(\mathbf{y}_i) L_{\boldsymbol{\xi}}(\mathbf{y}_{i+1}) \ll \|\mathbf{y}_i\|^{\rho-2+\mathcal{O}_n(\delta)},$$

$$\|\mathbf{z}\| \ll \|\mathbf{y}_i\|^2 L_{\boldsymbol{\xi}}(\mathbf{y}_{i+1}) + \|\mathbf{y}_i\|^{\rho-2+\mathcal{O}_n(\delta)} \ll \|\mathbf{y}_i\|^{2-\rho+\mathcal{O}_n(\delta)},$$

with implied constants that are independent of i. Since $2 - \rho = 1/\rho^n$, this means that, for $j = i - n$, we have

$$\|\mathbf{z}\| \leq \|\mathbf{y}_j\|^{1+\mathcal{O}_n(\delta)} \quad \text{and} \quad L_{\boldsymbol{\xi}}(\mathbf{z}) \leq \|\mathbf{y}_j\|^{-1+\mathcal{O}_n(\delta)} = L_{\boldsymbol{\xi}}(\mathbf{y}_j)^{1+\mathcal{O}_n(\delta)}.$$

Now, suppose that $\mathbf{z}$ is not a multiple of $\mathbf{y}_j$. Then $\mathbf{z}$ and $\mathbf{y}_j$ are linearly independent over $\mathbb{Q}$ and so there exists an integer k with $1 \leq k \leq n$ such that $\mathbf{y}_{j-n}, \ldots, \widehat{\mathbf{y}_{j-k}}, \ldots, \mathbf{y}_j, \mathbf{z}$ form a basis of $\mathbb{Q}^{n+1}$. Since $\mathbf{y}_j$ has the largest norm and yields the smallest value for the function $L_{\boldsymbol{\xi}}$ among the points $\mathbf{y}_{j-n}, \ldots, \mathbf{y}_j$, we find

$$1 \leq |\det(\mathbf{y}_{j-n}, \ldots, \widehat{\mathbf{y}_{j-k}}, \ldots, \mathbf{y}_j, \mathbf{z})|$$
$$\ll \left(\|\mathbf{z}\| L_{\boldsymbol{\xi}}(\mathbf{y}_j) + L_{\boldsymbol{\xi}}(\mathbf{z})\|\mathbf{y}_j\|\right) L_{\boldsymbol{\xi}}(\mathbf{y}_{j-n}) \cdots \widehat{L_{\boldsymbol{\xi}}(\mathbf{y}_{j-k})} \cdots L_{\boldsymbol{\xi}}(\mathbf{y}_{j-1})$$
$$\ll \|\mathbf{y}_j\|^{\mathcal{O}_n(\delta)-(\rho^{-n}+\cdots+\rho^{-2})}.$$

Assuming δ small enough as a function of n alone, this yields an upper bound on $\|\mathbf{y}_j\|$ and thus on i. Then Condition (iv) is fulfilled as well. $\qquad\square$

8. Construction of extremal points

We now turn to the proof of Theorem 2.2 (ii), so $n \geq 2$. As mentioned after Lemma 4.1, we may assume that the hypersurface Z of $\mathbb{P}^{n+1}(\mathbb{R})$ is defined by the quadratic form $q \colon \mathbb{Q}^{n+1} \to \mathbb{Q}$ given by

$$(8.1) \qquad q(t_0, \ldots, t_n) = t_0^2 - a_1 t_1^2 - \cdots - a_n t_n^2$$

for integers $a_1, \ldots, a_n$ where $a_1 > 0$ is not a square, or by

$$(8.2) \qquad q(t_0, \ldots, t_n) = t_0 t_1 - a_2 t_2^2 - \cdots - a_n t_n^2$$

for integers $a_2, \ldots, a_n$ with $a_2 \neq 0$. We need to show the existence of infinitely many points ξ in Z^{li} such that

$$(8.3) \qquad \limsup_{X \to \infty} X^{1/\rho} \mathcal{D}_\xi(X; E) < \infty$$

where $E = \mathbb{P}^n(\mathbb{Q}) \setminus Z$ and where $\rho = \rho_n$ is as in the statement of Theorem 1.1. However, showing the existence of a single point ξ suffices because if ξ has this property, then it follows from Lemma 4.1 that $T(\xi)$ shares the same property for any automorphism $T \in \mathrm{GL}_{n+1}(\mathbb{Q})$ such that $q \circ T = q$. Indeed, this group of automorphisms of q is infinite and we have $T(\xi) = \xi$ if and only if $T = \pm I$. For example, if q is given by (8.1), we obtain an automorphism T of infinite order by choosing a solution $(u, v) \in \mathbb{Z}^2$ of the Pell equation $u^2 - a_1 v^2 = 1$ with $v \neq 0$ and by defining

$$T(t_0, t_1, \ldots, t_n) = (u t_0 + a_1 v t_1, v t_0 + u t_1, t_2, \ldots, t_n)$$

(this corresponds to multiplication by $u + v\sqrt{a_1}$ in $\mathbb{Q}(\sqrt{a_1})$ via the natural isomorphism between $\mathbb{Q}^{n+1}$ and $\mathbb{Q}(\sqrt{a_1}) \times \mathbb{Q}^{n-1}$). If q is given by (8.2), the automorphism T given by

$$T(t_0, t_1, \ldots, t_n) = (2 t_0, t_1/2, t_2, \ldots, t_n)$$

is also of infinite order.

Thus we simply need to construct one point $\xi \in Z^{\mathrm{li}}$ with the property (8.3). We achieve this through the following result.

Theorem 8.1. *There exists an unbounded sequence of points $(\mathbf{x}_i)_{i \geq 0}$ in $\mathbb{Z}^{n+1}$ which upon setting $X_i = \|\mathbf{x}_i\|$ satisfies for each $i \geq 0$*

    (i) $\mathbf{x}_{i+n+1} = \psi(\mathbf{x}_{i+n}, \mathbf{x}_i)$,
    (ii) $q(\mathbf{x}_i) = 1$,
    (iii) $|\det(\mathbf{x}_i, \ldots, \mathbf{x}_{i+n})| = |\det(\mathbf{x}_0, \ldots, \mathbf{x}_n)| \neq 0$,
    (iv) $X_{i+1} \asymp X_i^\rho$,
    (v) $\|\mathbf{x}_{i+1} \wedge \mathbf{x}_i\| \asymp X_{i+1}/X_i$,

with implied constants that are independent of i. Its image $\big([\mathbf{x}_i]\big)_{i \geq 0}$ in $\mathbb{P}^n(\mathbb{R})$ converges to a point ξ in Z^{li} such that $D_\xi(\mathbf{x}_i) \asymp X_i^{-1}$. Moreover, this point ξ satisfies (8.3).

Arguing as in the proof of Proposition 7.3, one may show that, except for the very precise form of Conditions (i) to (iii), the existence of such a sequence $(\mathbf{x}_i)_{i \geq 0}$ is forced upon if we assume that a point $\xi \in Z^{\mathrm{li}}$ satisfies (8.3). The theorem shows the converse.

The proof of Theorem 8.1 requires two main steps. We first construct linearly independent points $\mathbf{x}_0, \ldots, \mathbf{x}_n$ from the set

$$U = \{\mathbf{x} \in \mathbb{Z}^{n+1} \, ; \, q(\mathbf{x}) = 1\}.$$

Then we extend them into an infinite sequence $(\mathbf{x}_i)_{i \geq 0}$ using the recurrence relation (i). The resulting sequence is entirely contained in U because Lemma 3.1 shows that $\psi(\mathbf{x}, \mathbf{y}) \in U$ for any $\mathbf{x}, \mathbf{y} \in U$. So, the recurrence relation (i) simplifies to

$$(8.4) \qquad \mathbf{x}_{i+n+1} = b(\mathbf{x}_i, \mathbf{x}_{i+n})\mathbf{x}_{i+n} - \mathbf{x}_i \quad (i \geq 0)$$

where b denotes the symmetric bilinear form attached to q (characterized by (3.1)). In turn this implies that

$$|\det(\mathbf{x}_{i+1}, \ldots, \mathbf{x}_{i+n+1})| = |\det(\mathbf{x}_i, \ldots, \mathbf{x}_{i+n})|$$

for each $i \geq 0$. Thus Conditions (ii) and (iii) are automatically satisfied. We show that for a suitable choice of $\mathbf{x}_0, \ldots, \mathbf{x}_n$, the asymptotic estimates (iv) and (v) hold as well, thus proving the first assertion of the theorem. As we will see, the second assertion follows easily from this.

Before, we go on with the proof, we note that the polynomial

$$h(x) = (x - 1)(x^n - x^{n-1} - \cdots - x - 1) = x^{n+1} - 2x^n + 1$$

has only two positive real roots, 1 and $\rho = \rho_n \in (1, 2)$. Moreover, for any sufficiently small $\epsilon > 0$, we have $2(1 + \epsilon)^n > (1 + \epsilon)^{n+1} + 1$ and so $|2x^n| > |x^{n+1} + 1|$ for each $x \in \mathbb{C}$ with $|x| = 1 + \epsilon$. By Rouché's theorem, this means that $h(x)$ has the same number n of complex roots as $2x^n$ in the closed disk $|x| \leq 1 + \epsilon$. Since ϵ can be taken arbitrarily small, and since $x = 1$ is the only root of h with $|x| = 1$, we conclude that besides 1 and ρ, all roots x of h have $|x| < 1$. In particular, the algebraic integer ρ is a Pisot number.

Step 1: Choice of initial points.

Proposition 8.2. *There exist a constant $C_0 > 1$ and points $\mathbf{x}_0, \ldots, \mathbf{x}_{n-1} \in U$ with the following property. For each $B \geq C_0$, there exists $\mathbf{x}_n \in U$ such that*

$$(8.5) \qquad \|\mathbf{x}_n\|, \ |\det(\mathbf{x}_0, \mathbf{x}_1, \ldots, \mathbf{x}_n)|, \ |b(\mathbf{x}_i, \mathbf{x}_n)|, \ \|\mathbf{x}_i \wedge \mathbf{x}_n\| \in \left[\frac{B}{C_0}, \, C_0 B\right]$$

for $i = 1, \ldots, n - 1$.

The most delicate case is when the quadratic form is given by (8.1). The proof then relies on a theorem of Lagrange saying that, for any positive integer a, the Pell equation $x^2 - ay^2 = 1$ admits infinitely many solutions $(x, y) \in \mathbb{Z}^2$ if and only if a is not a square. We will also need the following consequence of that result.

Lemma 8.3. *Let $a, b \in \mathbb{Z}$ with $a > 0$. Then the equation $x^2 - ay^2 - bz^2 = 1$ admits at least one solution $(x, y, z) \in \mathbb{Z}^3$ with $xz \neq 0$.*

Proof. If $b = 0$, we have the solution $(x, y, z) = (1, 0, 1)$. Suppose that $b \neq 0$. Then the polynomial $at^2 + b \in \mathbb{Z}[t]$ is not the square of a polynomial of $\mathbb{Z}[t]$ and so by a statement of Pólya and Szegö [11, Problem 114, p. 132], there are arbitrarily large integers m for which $p(m) = am^2 + b$ is not a square (see [9] for generalizations). In the present case, one can even show that, for any sufficiently large integer k, at least one of the integers $ak^2 + b$ or $a(k+1)^2 + b$ is not a square. Fix $m \in \mathbb{Z}$ such that $c = am^2 + b$ is positive but not a square, and put $y = mz$. Then the equation becomes $x^2 - cz^2 = 1$ which, by the theorem of Lagrange, admits a solution $(x, z) \in \mathbb{Z}^2$ with $xz \neq 0$. Then (x, mz, z) is a solution of the original equation with the requested property. $\square$

Proof of Proposition 8.2. For $i = 0, 1, \ldots, n$, let $\mathbf{e}_i$ denote the point of $\mathbb{Z}^{n+1}$ whose i-th coordinate is 1 and all other coordinates are 0.

Suppose first that q is given by (8.1) with $a_1, \ldots, a_n \in \mathbb{Z}$ and $a_1 > 0$ non-square. Then, for $i = 0, \ldots, n-2$, Lemma 8.3 ensures the existence of a point $\mathbf{x}_i$ in U of the form

$$\mathbf{x}_i = k_i \mathbf{e}_0 + \ell_i \mathbf{e}_1 + m_i \mathbf{e}_{i+2}$$

with $(k_i, \ell_i, m_i) \in \mathbb{Z}^3$ and $k_i m_i \neq 0$. By Lagrange's theorem, there exists also an infinite set of points of U of the form

$$\mathbf{x}_{n-1} = k_{n-1} \mathbf{e}_0 + \ell_{n-1} \mathbf{e}_1$$

with integers $k_{n-1}, \ell_{n-1} \geq 1$. Fix such a choice of $\mathbf{x}_0, \ldots, \mathbf{x}_{n-1} \in U$ and put

$$\mathbf{x}_n = k \mathbf{e}_0 + \ell \mathbf{e}_1$$

for another pair (k, ℓ) of positive integers satisfying $k^2 - a_1 \ell^2 = 1$ with $\ell \neq \ell_{n-1}$. Then $\mathbf{x}_0, \ldots, \mathbf{x}_n \in U$ are linearly independent over $\mathbb{Q}$. Since $k = \sqrt{a_1}\ell + \mathcal{O}(1/\ell)$, we find

$$\|\mathbf{x}_n\| \asymp |\det(\mathbf{x}_0, \mathbf{x}_1, \ldots, \mathbf{x}_n)| \asymp \ell,$$
$$\ell \gg |b(\mathbf{x}_i, \mathbf{x}_n)| = 2|kk_i - a_1\ell\ell_i| = 2\sqrt{a_1}\ell|k_i - \sqrt{a_1}\ell_i| + \mathcal{O}(1/\ell) \gg \ell,$$
$$\ell \gg \|\mathbf{x}_i \wedge \mathbf{x}_n\| \geq |k\ell_i - \ell k_i| = \ell|k_i - \sqrt{a_1}\ell_i| + \mathcal{O}(1/\ell) \gg \ell$$

for $i = 0, \ldots, n-1$. The conclusion follows because the admissible values of ℓ have exponential growth (they form a linear recurrence sequence).

Suppose now that q is given by (8.2) with $a_2, \ldots, a_{n-1} \in \mathbb{Z}$ and $a_2 \neq 0$. For a given integer $\ell \geq 2$, we choose

$$\mathbf{x}_i = \begin{cases} (a_{i+2} + 1)\mathbf{e}_0 + \mathbf{e}_1 + \mathbf{e}_{i+2} & \text{for } i = 0, \ldots, n-2, \\ \mathbf{e}_0 + \mathbf{e}_1 & \text{if } i = n-1, \\ (a_2\ell^2 + 1)\mathbf{e}_0 + \mathbf{e}_1 + \ell\mathbf{e}_2 & \text{if } i = n. \end{cases}$$

Then $\mathbf{x}_0, \ldots, \mathbf{x}_n \in U$ are linearly independent over $\mathbb{Q}$. Moreover, as functions of ℓ, the numbers $\|\mathbf{x}_n\|$, $|\det(\mathbf{x}_0, \mathbf{x}_1, \ldots, \mathbf{x}_n)|$, $|b(\mathbf{x}_i, \mathbf{x}_n)|$ and $\|\mathbf{x}_i \wedge \mathbf{x}_n\|$ for $i = 0, \ldots, n-1$ are all equal to $|a_2|\ell^2 + \mathcal{O}(\ell)$ or $\sqrt{2}|a_2|\ell^2 + \mathcal{O}(\ell)$. The conclusion follows by varying ℓ. $\qquad\square$

Step 2: Asymptotic estimates. We will use the approximation lemma of Appendix A to prove the following statement.

Proposition 8.4. *Let C_0 and $\mathbf{x}_0, \ldots, \mathbf{x}_{n-1} \in U$ be as in Proposition 8.2 for the given quadratic form q. For each sufficiently large B with $B \geq C_0$, the point $\mathbf{x}_n \in U$ provided by Proposition 8.2 has the following property. The sequence $(\mathbf{x}_i)_{i \geq 0}$ built on $(\mathbf{x}_0, \ldots, \mathbf{x}_n)$ using the recurrence formula*

$$(8.6) \qquad \mathbf{x}_{i+1} = \psi(\mathbf{x}_i, \mathbf{x}_{i-n}) = b(\mathbf{x}_{i-n}, \mathbf{x}_i)\mathbf{x}_i - \mathbf{x}_{i-n} \quad (i \geq n).$$

is contained in U and satisfies

$$\| \mathbf{x}_{i+1} \| \asymp \| \mathbf{x}_i \|^\rho \quad and \quad \| \mathbf{x}_i \wedge \mathbf{x}_{i+1} \| \asymp \frac{\| \mathbf{x}_{i+1} \|}{\| \mathbf{x}_i \|}.$$

Proof. Fix a choice of $B \geq C_0$ and of a corresponding point $\mathbf{x}_n \in U$ as in Proposition 8.2, and consider the sequence $(\mathbf{x}_i)_{i \geq 0}$ given by (8.6). By Lemma 3.1, this sequence is contained in U. By linearity, the formula (8.6) yields

$$b(\mathbf{x}_{i+1-j}, \mathbf{x}_{i+1}) = b(\mathbf{x}_{i-n}, \mathbf{x}_i)b(\mathbf{x}_{i+1-j}, \mathbf{x}_i) - b(\mathbf{x}_{i-n}, \mathbf{x}_{i+1-j}),$$

$$\mathbf{x}_{i+1-j} \wedge \mathbf{x}_{i+1} = b(\mathbf{x}_{i-n}, \mathbf{x}_i)\mathbf{x}_{i+1-j} \wedge \mathbf{x}_i + \mathbf{x}_{i-n} \wedge \mathbf{x}_{i+1-j}$$

for each choice of integers i and j with $1 \leq j \leq n \leq i$. Putting

$$b_i^{(j)} = b(\mathbf{x}_{i-j}, \mathbf{x}_i) \quad \text{and} \quad \mathbf{y}_i^{(j)} = \mathbf{x}_{i-j} \wedge \mathbf{x}_i$$

these equalities become

$$(8.7) \qquad b_{i+1}^{(j)} = b_i^{(n)} b_i^{(j-1)} - b_{i+1-j}^{(n+1-j)} \quad \text{and} \quad \mathbf{y}_{i+1}^{(j)} = b_i^{(n)} \mathbf{y}_i^{(j-1)} + \mathbf{y}_{i+1-j}^{(n+1-j)} \quad (1 \leq j \leq n \leq i).$$

In particular, for $j = 1$, they simplify to

$$(8.8) \qquad b_{i+1}^{(1)} = b_i^{(n)} \quad \text{and} \quad \mathbf{y}_{i+1}^{(1)} = \mathbf{y}_i^{(n)} \quad (i \geq n),$$

since $b_i^{(0)} = b(\mathbf{x}_i, \mathbf{x}_i) = 2q(\mathbf{x}_i) = 2$ and $\mathbf{y}_i^{(0)} = \mathbf{x}_i \wedge \mathbf{x}_i = 0$. Put also

$$B_i = \max\{|b_k^{(j)}| \,;\, 0 \leq k \leq i \ \text{ and } \ 0 \leq j \leq \min\{n, k\}\},$$

$$Y_i = \max\{\|\mathbf{y}_k^{(j)}\| \,;\, 0 \leq k \leq i \ \text{ and } \ 0 \leq j \leq \min\{n, k\}\},$$

$$X_i = \max\{\|\mathbf{x}_k\| \,;\, 0 \leq k \leq i\}$$

for each $i \geq 0$. For $i \geq n$ and $j \in \{2, \ldots, n\}$, the formulas (8.7) imply that

$$(8.9) \qquad |b_i^{(n)}| \, |b_i^{(j-1)}| - B_{i-1} \leq |b_{i+1}^{(j)}| \leq |b_i^{(n)}| \, |b_i^{(j-1)}| + B_{i-1},$$

$$(8.10) \qquad |b_i^{(n)}| \, \|\mathbf{y}_i^{(j-1)}\| - Y_{i-1} \leq \|\mathbf{y}_{i+1}^{(j)}\| \leq |b_i^{(n)}| \, \|\mathbf{y}_i^{(j-1)}\| + Y_{i-1},$$

while the recurrence formula (8.6) yields

$$\text{(8.11)} \qquad |b_i^{(n)}| \, \| \mathbf{x}_i \| - X_{i-1} \leq \| \mathbf{x}_{i+1} \| \leq |b_i^{(n)}| \, \| \mathbf{x}_i \| + X_{i-1}.$$

We first show that, if B is large enough, the numbers $|b_i^{(j)}|$ and $\| \mathbf{y}_i^{(j)} \|$ behave like powers of B. To this end, consider the integers $m(i, j)$ defined recursively for $i \geq 0$ and $j = 1, \ldots, n$ by

$$m(i, j) = \begin{cases} 0 & \text{if } 0 \leq i < n \text{ and } 1 \leq j \leq n, \\ 1 & \text{if } i = n \text{ and } 1 \leq j \leq n, \\ m(i-1, n) & \text{if } i > n \text{ and } j = 1, \\ m(i-1, n) + m(i-1, j-1) & \text{if } i > n \text{ and } 2 \leq j \leq n. \end{cases}$$

It is relatively easy to prove that they satisfy

$$\text{(8.12)} \qquad m(i, 1) \leq m(i, 2) \leq \cdots \leq m(i, n) \quad (i \geq 0),$$

$$\text{(8.13)} \qquad \frac{3}{2} m(i, n) \leq m(i+1, n) \leq 2m(i, n) \quad (i \geq n).$$

In particular $m(i, n)$ tends to infinity with i. Define also

$$C_n = \max\{C_0, B_{n-1}, X_{n-1}, Y_{n-1}\} \quad \text{and} \quad C_{i+1} = 2^{1/m(i,n)} C_i \quad (i \geq n).$$

These numbers are independent of B since B_{n-1}, X_{n-1} and Y_{n-1} are functions of $\mathbf{x}_0, \ldots, \mathbf{x}_{n-1}$ only. By (8.13), we have $m(i, n) \geq (3/2)^{i-n}$ for each $i \geq n$, thus $\sum_{i=n}^{\infty} 1/m(i, n) \leq 3$ and so the sequence $(C_i)_{i \geq n}$ is bounded above by $C := 8C_n$.

Assume from now on that $B \geq C^5$. We claim that for each $i \geq n$ we have

$$\text{(8.14)} \qquad X_i = \| \mathbf{x}_i \| > X_{i-1},$$

$$\text{(8.15)} \qquad \max\{B_{i-1}, Y_{i-1}\} \leq \frac{1}{2} \left(\frac{B}{C_i} \right)^{m(i,n)},$$

$$\text{(8.16)} \qquad \min\{|b_i^{(j)}|, \| \mathbf{y}_i^{(j)} \|\} \geq \left(\frac{B}{C_i} \right)^{m(i,j)} \quad (1 \leq j \leq n),$$

$$\text{(8.17)} \qquad \max\{|b_i^{(j)}|, \| \mathbf{y}_i^{(j)} \|\} \leq (C_i B)^{m(i,j)} \quad (1 \leq j \leq n).$$

Arguing by induction on i, suppose first that $i = n$. Then we have $m(i, j) = 1$ for all j. So (8.16) and (8.17) follow immediately from the choice of $\mathbf{x}_n$ (this only requires $B \geq C_0$). Since $\| \mathbf{x}_n \| \geq B/C_0$ and $X_{n-1} \leq C_n$, we also have $\| \mathbf{x}_n \| > X_{n-1}$, as $B > C_0 C_n$. This yields (8.14). Finally (8.15) follows from $\max\{B_{n-1}, Y_{n-1}\} \leq C_n$, as $B \geq 2C_n^2$.

Suppose now that (8.14) to (8.17) hold for some integer $i \geq n$. Then (8.16) gives $|b_i^{(n)}| \geq B/C_i \geq 2$. Thus (8.11) together with (8.14) yield

$$\| \mathbf{x}_{i+1} \| \geq 2X_i - X_{i-1} > X_i,$$

so $X_{i+1} = \| \mathbf{x}_{i+1} \| > X_i$. Because of (8.12) and (8.15), the inequalities (8.17) imply

$$\text{(8.18)} \qquad \max\{B_i, Y_i\} \leq (C_i B)^{m(i,n)}.$$

Since $m(i+1,n) \geq (3/2)m(i,n)$ and $B \geq C_{i+1}^5$, we deduce that

$$\Big(\frac{B}{C_{i+1}}\Big)^{m(i+1,n)} \geq \Big(\frac{B}{C_{i+1}^5}\Big)^{m(i,n)/2}(C_{i+1}B)^{m(i,n)} \geq 2\max\{B_i, Y_i\},$$

which proves (8.15) with i replaced by $i+1$. Using the hypothesis (8.16), the equalities (8.8) yield

$$\min\{|b_{i+1}^{(1)}|, \|\mathbf{y}_{i+1}^{(1)}\|\} = \min\{|b_i^{(n)}|, \|\mathbf{y}_i^{(n)}\|\} \geq \Big(\frac{B}{C_i}\Big)^{m(i,n)} \geq \Big(\frac{B}{C_{i+1}}\Big)^{m(i+1,1)},$$

while using (8.15) and (8.16), the estimates (8.9) and (8.10) imply for $j = 2,\ldots,n$

$$\min\{|b_{i+1}^{(j)}|, \|\mathbf{y}_{i+1}^{(j)}\|\} \geq |b_i^{(n)}|\min\{|b_i^{(j-1)}|, \|\mathbf{y}_i^{(j-1)}\|\} - \max\{B_{i-1}, Y_{i-1}\}$$
$$\geq \Big(\frac{B}{C_i}\Big)^{m(i,n)+m(i,j-1)} - \frac{1}{2}\Big(\frac{B}{C_i}\Big)^{m(i,n)}$$
$$\geq \frac{1}{2}\Big(\frac{B}{C_i}\Big)^{m(i+1,j)}$$
$$\geq \Big(\frac{B}{C_{i+1}}\Big)^{m(i+1,j)}.$$

This proves (8.16) with i replaced by $i+1$. We omit the proof of the induction step for (8.17) as it is similar. In that case, we may even replace the use of (8.15) by the weaker estimate $\max\{B_{i-1}, Y_{i-1}\} \leq (C_i B)^{m(i,n)}$ which follows from (8.18).

So, under the current hypothesis $B \geq C^5$, we have

$$X_i = \|\mathbf{x}_i\| > X_{i-1}, \quad \max\{B_{i-1}, Y_{i-1}\} \leq \frac{1}{2}\Big(\frac{B}{C_i}\Big)^{m(i,n)}, \quad \min\{|b_i^{(j)}|, \|\mathbf{y}_i^{(j)}\|\} \geq \Big(\frac{B}{C_i}\Big)^{m(i,j)}$$

whenever $1 \leq j \leq n \leq i$. In particular, this yields the crude estimate

$$\max\{B_{i-1}, Y_{i-1}\} \leq |b_i^{(n)}|.$$

Thus, for each $i \geq n$ and $j = 2,\ldots,n$, the inequalities (8.9), (8.10) and (8.11) imply that

$$|b_{i+1}^{(j)}| = |b_i^{(n)}|\,|b_i^{(j-1)}|(1 + \epsilon_{i,j}) \qquad \text{with} \quad |\epsilon_{i,j}| \leq \frac{B_{i-1}}{|b_i^{(n)}|\,|b_i^{(j-1)}|} \leq \Big(\frac{C}{B}\Big)^{m(i,j-1)},$$

$$\|\mathbf{y}_{i+1}^{(j)}\| = |b_i^{(n)}|\,\|\mathbf{y}_i^{(j-1)}\|(1 + \epsilon_{i,j}') \qquad \text{with} \quad |\epsilon_{i,j}'| \leq \frac{Y_{i-1}}{|b_i^{(n)}|\,\|\mathbf{y}_i^{(j-1)}\|} \leq \Big(\frac{C}{B}\Big)^{m(i,j-1)},$$

$$X_{i+1} = |b_i^{(n)}|\,X_i(1 + \epsilon_i'') \qquad \text{with} \quad |\epsilon_i''| \leq \frac{X_{i-1}}{|b_i^{(n)}|\,X_i} \leq \Big(\frac{C}{B}\Big)^{m(i,n)}.$$

For $j = 1$, we have instead the formulas (8.8). We conclude that, for each $i \geq n$, the vector

$$\mathbf{v}_i := \big(\log|b_i^{(1)}|, \ldots, \log|b_i^{(n)}|, \log\|\mathbf{y}_i^{(1)}\|, \ldots, \log\|\mathbf{y}_i^{(n)}\|, \log X_i\big) \in \mathbb{R}^{2n+1},$$

satisfies

$$\|\mathbf{v}_{i+1} - T(\mathbf{v}_i)\| \ll \Big(\frac{C}{B}\Big)^{m(i,1)},$$

where $T\colon \mathbb{R}^{2n+1} \to \mathbb{R}^{2n+1}$ is the linear operator given by

$$T(x_1, \ldots, x_n, y_1, \ldots, y_n, z)$$
$$= (x_n, x_1 + x_n, \ldots, x_{n-1} + x_n, y_n, y_1 + x_n, \ldots, y_{n-1} + x_n, z + x_n).$$

We note that the matrix of T in the canonical basis is lower triangular by blocks with three blocks on the diagonal, namely the companion matrices of the polynomials

$$p(x) = x^n - x^{n-1} - \cdots - x - 1, \quad q(x) = x^n - 1 \quad \text{and} \quad r(x) = x - 1.$$

Thus the characteristic polynomial of T is $p(x)q(x)r(x)$. We also observe that $x = 1$ is the only multiple root of this product and that it is a double root (this follows for example from $p(x)r(x) = x^{n+1} - 2x^n + 1 = (x-2)q(x) + (x-1)$). On the other hand, the eigenspace of T for the eigenvalue 1 contains the vectors

$$\big(\underbrace{0, \ldots, 0}_{n}, \underbrace{1, \ldots, 1}_{n}, 0\big) \quad \text{and} \quad \big(\underbrace{0, \ldots, 0}_{n}, \underbrace{0, \ldots, 0}_{n}, 1\big).$$

So it has dimension 2 and thus the minimal polynomial of T must be $m_T(x) = p(x)q(x)$. Its roots in $\mathbb{C}$ are all simple and, by the comments made before Step 1, its root $\rho = \rho_n$ is the only one of absolute value > 1. Moreover

$$\mathbf{v} = \Big(1 - \frac{1}{\rho}, \ldots, 1 - \frac{1}{\rho^n}, 1 - \frac{1}{\rho}, \ldots, 1 - \frac{1}{\rho^n}, 1\Big)$$

is an eigenvector of T for $\rho = 2 - 1/\rho^n$. As $\sum_i (C/B)^{m(i,1)} < \infty$, the approximation lemma A.1 in the appendix yields a constant $\alpha \in \mathbb{R}$ such that the differences $\mathbf{v}_i - \alpha \rho^i \mathbf{v}$ are bounded. This means that for each pair of integers (i, j) with $1 \le j \le n \le i$ we have

$$|b_i^{(j)}| \asymp \| \mathbf{y}_i^{(j)} \| \asymp \exp(\alpha(\rho^i - \rho^{i-j})) \quad \text{and} \quad X_i \asymp \exp(\alpha \rho^i),$$

so $X_{i+1} \asymp X_i^\rho$ and $\| \mathbf{x}_i \wedge \mathbf{x}_{i+1} \| = \| \mathbf{y}_{i+1}^{(1)} \| \asymp X_{i+1}/X_i$. $\qquad \square$

Proof of Theorem 8.1. Proposition 8.4 provides a sequence $(\mathbf{x}_i)_{i \ge 0}$ in U which satisfies the five conditions (i) to (v) of the theorem. By (v), we have

$$\operatorname{dist}([\mathbf{x}_i], [\mathbf{x}_{i+1}]) = \frac{\| \mathbf{x}_i \wedge \mathbf{x}_{i+1} \|}{X_i X_{i+1}} \asymp X_i^{-2},$$

and, by (iv), X_i goes to infinity faster than any geometric sequence. Thus $[\mathbf{x}_i]$ converges to a point $\xi \in \mathbb{P}^n(\mathbb{R})$ with

$$D_\xi(\mathbf{x}_i) = X_i \operatorname{dist}([\mathbf{x}_i], \xi) \asymp X_i^{-1}.$$

Choose a representative $\boldsymbol{\xi} \in \mathbb{R}^{n+1}$ with $\| \boldsymbol{\xi} \| = 1$. As $\pm X_i^{-1} \mathbf{x}_i$ converges to $\boldsymbol{\xi}$ for an appropriate choice of signs, and as $q(\pm X_i^{-1} \mathbf{x}_i) = X_i^{-2}$ converges to 0, we find that $q(\boldsymbol{\xi}) = 0$, thus $\xi \in Z$. By Lemma 6.2, we deduce that $\xi \in Z^{\mathrm{li}}$ because we have $\lim_{i \to \infty} D_\xi(\mathbf{x}_i) = 0$ and the crucial property (iii) gives $\langle \mathbf{x}_i, \ldots, \mathbf{x}_{i+n} \rangle_\mathbb{Q} = \mathbb{Q}^{n+1}$ for each $i \ge 0$. Finally, for each sufficiently large X, there exists an index $i \ge 0$ such that $X_i \le X < X_{i+1}$ and, since $[\mathbf{x}_i] \in E$, we obtain

$$X^{1/\rho} \mathcal{D}_\xi(X; E) \ll X^{1/\rho} D_\xi(\mathbf{x}_i) \ll X_{i+1}^{1/\rho} X_i^{-1} \ll 1,$$

showing that $\limsup_{X \to \infty} X^{1/\rho} \mathcal{D}_\xi(X; E) < \infty$, as announced.

9. Quadratic forms of Witt index at most 1

In this section, we simply assume that $n \geq 1$ and we fix a quadratic form q on $\mathbb{Q}^{n+1}$. Our goal is to prove Theorem 2.2 (iv). We start with a general estimate which compares to [4, Lemme Clef] for isotropic subspaces of dimension 1.

Lemma 9.1. *Suppose that $\mathbf{x}$ and $\mathbf{y}$ are linearly independent points of $\mathbb{Q}^{n+1}$, that the subspace W of $\mathbb{Q}^{n+1}$ that they span is not totally isotropic, and that $q(\mathbf{y}) = 0$. Put $\mathbf{z} = \psi(\mathbf{x}, \mathbf{y})$. Then $\mathbf{z}$ is non-zero with $q(\mathbf{z}) = 0$, and we have*

$$\|\mathbf{y}\| \, \|\mathbf{z}\| \leq 2\|q\| \, \|\mathbf{x} \wedge \mathbf{y}\|^2$$

where $\|q\| := \max\{|q_\mathbb{R}(\mathbf{x})| \, ; \, \|\mathbf{x}\| = 1\}$.

Proof. As $\{\mathbf{x}, \mathbf{y}\}$ is a basis of W and as W is not totally isotropic, we have $b(\mathbf{x}, \mathbf{y}) \neq 0$ or $q(\mathbf{x}) \neq 0$, and so $\mathbf{z} = b(\mathbf{x}, \mathbf{y})\mathbf{x} - q(\mathbf{x})\mathbf{y}$ is non-zero. We also have $q(\mathbf{z}) = q(\mathbf{x})^2 q(\mathbf{y}) = 0$ by Lemma 3.1. Since the angle between the lines spanned by $\mathbf{y}$ and $\mathbf{z}$ in $\mathbb{R}^{n+1}$ is at most $\pi/2$, there is a vector $\mathbf{u} \in \mathbb{R}^{n+1}$ of norm 1 which makes angles of at most $\pi/4$ with each of those lines. This means that

$$|\mathbf{u} \cdot \mathbf{y}| \geq \frac{1}{\sqrt{2}} \|\mathbf{y}\| \quad \text{and} \quad |\mathbf{u} \cdot \mathbf{z}| \geq \frac{1}{\sqrt{2}} \|\mathbf{z}\|$$

where the dot represents the standard scalar product in $\mathbb{R}^{n+1}$. The point

$$\mathbf{w} := (\mathbf{u} \cdot \mathbf{y})\mathbf{x} - (\mathbf{u} \cdot \mathbf{x})\mathbf{y}$$

obtained by contraction of $\mathbf{x} \wedge \mathbf{y}$ with $\mathbf{u}$ has norm $\|\mathbf{w}\| \leq \|\mathbf{u}\| \, \|\mathbf{x} \wedge \mathbf{y}\| = \|\mathbf{x} \wedge \mathbf{y}\|$ by (6.4). Since $q(\mathbf{y}) = 0$, we find that

$$q_\mathbb{R}(\mathbf{w}) = (\mathbf{u} \cdot \mathbf{y})^2 q(\mathbf{x}) - (\mathbf{u} \cdot \mathbf{x})(\mathbf{u} \cdot \mathbf{y})b(\mathbf{x}, \mathbf{y}) = -(\mathbf{u} \cdot \mathbf{y})(\mathbf{u} \cdot \mathbf{z}).$$

Altogether, this yields

$$\frac{1}{2}\|\mathbf{y}\| \, \|\mathbf{z}\| \leq |q_\mathbb{R}(\mathbf{w})| \leq \|q\| \, \|\mathbf{w}\|^2 \leq \|q\| \, \|\mathbf{x} \wedge \mathbf{y}\|^2. \qquad \square$$

Corollary 9.2. *Suppose further that $\mathbf{x}, \mathbf{y} \in \mathbb{Z}^{n+1}$, then we have $\|\mathbf{y}\| \leq c\|\mathbf{x} \wedge \mathbf{y}\|^2$ for a constant $c > 0$ depending only on q.*

Proof. Choose an integer $m \geq 1$ such that $mq(\mathbb{Z}^{n+1}) \subseteq \mathbb{Z}$. Then $m\mathbf{z}$ is a non-zero integer point, so $\|\mathbf{z}\| \geq 1/m$ and therefore $\|\mathbf{y}\| \leq 2m\|q\|\|\mathbf{x} \wedge \mathbf{y}\|^2$. $\qquad \square$

We can now state and prove the main result of this section. In a corollary below, we will show that it implies Theorem 2.2 (iv).

Proposition 9.3. *Let $Z = Z(q_\mathbb{R}) \subseteq \mathbb{P}^n(\mathbb{R})$. Suppose that q has Witt index $m \leq 1$ and that a point $\xi \in Z^{\mathrm{li}}$ has $\hat{\lambda}(\xi) > 1/2$. Then, we have $D_\xi(\mathbf{x}) \gg \|\mathbf{x}\|^{-1/2}$ for each non-zero $\mathbf{x} \in \mathbb{Z}^{n+1}$ with $q(\mathbf{x}) = 0$.*

Proof. Suppose first that q is non-degenerate. Then, the maximal totally isotropic subspaces of $\mathbb{Q}^{n+1}$ have dimension $m \leq 1$. Choose a representative $\boldsymbol{\xi}$ of ξ in $\mathbb{R}^{n+1}$ with $\|\boldsymbol{\xi}\| = 1$, a real number λ with $1/2 < \lambda < \hat{\lambda}(\xi)$, and $X_0 \geq 1$ such that $\mathcal{D}_\xi(X) \leq X^{-\lambda}$ for each $X \geq X_0$. Suppose that $\mathbf{y}$ is a primitive point of $\mathbb{Z}^{n+1}$ with $q(\mathbf{y}) = 0$ and $\|\mathbf{y}\| \geq 2X_0$. There exists a non-zero point $\mathbf{x} \in \mathbb{Z}^{n+1}$ with $\|\mathbf{x}\| \leq \|\mathbf{y}\|/2$ and $D_\xi(\mathbf{x}) = \|\mathbf{x} \wedge \boldsymbol{\xi}\| \leq (\|\mathbf{y}\|/2)^{-\lambda}$. Then $\langle \mathbf{x}, \mathbf{y} \rangle_\mathbb{Q}$ is a subspace of $\mathbb{Q}^{n+1}$ of dimension 2, and so it is not totally isotropic. Since $q(\mathbf{y}) = 0$, Corollary 9.2 gives

$$\|\mathbf{y}\| \leq c\|\mathbf{x} \wedge \mathbf{y}\|^2$$

with $c = c(q) > 0$. On the other hand, the triangle inequality $\mathrm{dist}(\mathbf{x}, \mathbf{y}) \leq \mathrm{dist}(\mathbf{x}, \boldsymbol{\xi}) + \mathrm{dist}(\mathbf{y}, \boldsymbol{\xi})$ yields

$$\|\mathbf{x} \wedge \mathbf{y}\| \leq \|\mathbf{x}\|\|\mathbf{y} \wedge \boldsymbol{\xi}\| + \|\mathbf{y}\|\|\mathbf{x} \wedge \boldsymbol{\xi}\| \leq \frac{1}{2}\|\mathbf{y}\|\|\mathbf{y} \wedge \boldsymbol{\xi}\| + 2^\lambda \|\mathbf{y}\|^{1-\lambda}.$$

Altogether, this implies that

$$\|\mathbf{y} \wedge \boldsymbol{\xi}\| \geq 2\big(c^{-1/2}\|\mathbf{y}\|^{1/2} - 2^\lambda\|\mathbf{y}\|^{1-\lambda}\big)\|\mathbf{y}\|^{-1}$$

and thus $D_\xi(\mathbf{y}) \geq c^{-1/2}\|\mathbf{y}\|^{-1/2}$ if $\|\mathbf{y}\|$ is large enough. We conclude that $D_\xi(\mathbf{y}) \gg \|\mathbf{y}\|^{-1/2}$ for any primitive point $\mathbf{y}$ of $\mathbb{Z}^{n+1}$, and thus for any non-zero $\mathbf{y} \in \mathbb{Z}^{n+1}$.

In general, q has rank $r + 1$ for an integer r with $1 \leq r \leq n$ (we have $r \geq 1$ since $Z^{\mathrm{li}} \neq \emptyset$). Choose a representative $\boldsymbol{\xi}$ of ξ in $\mathbb{R}^{n+1}$, and a $\mathbb{Q}$-linear automorphism T of $\mathbb{Q}^{n+1}$ such that $T(\mathbb{Z}^{n+1}) = \mathbb{Z}^{n+1}$ and

$$\{0\}^{r+1} \times \mathbb{Q}^{n-r} = T^{-1}(\ker(q)) = \ker(q \circ T).$$

Then there is a non-degenerate quadratic form $\tilde{q} \colon \mathbb{Q}^{r+1} \to \mathbb{Q}$ such that

$$(q \circ T)(x_0, \ldots, x_n) = \tilde{q}(x_0, \ldots, x_r)$$

for each $(x_0, \ldots, x_n) \in \mathbb{Q}^{n+1}$. Set $\boldsymbol{\eta} = T_\mathbb{R}^{-1}(\boldsymbol{\xi})$ and write $\boldsymbol{\eta} = (\eta_0, \ldots, \eta_n)$. Then $\boldsymbol{\eta}$ has linearly independent coordinates over $\mathbb{Q}$ and the point $\eta = [\boldsymbol{\eta}]$ satisfies $\hat{\lambda}(\eta) = \hat{\lambda}(\xi)$. Moreover $\tilde{\boldsymbol{\eta}} = (\eta_0, \ldots, \eta_r)$ is a zero of $\tilde{q}_\mathbb{R}$. Thus $\tilde{\eta} = [\tilde{\boldsymbol{\eta}}]$ belongs to $\tilde{Z}^{\mathrm{li}}$ where $\tilde{Z} = Z(\tilde{q}_\mathbb{R})$ is the quadratic hypersurface of $\mathbb{P}^r(\mathbb{R})$ associated to $\tilde{q}$. We also have $\hat{\lambda}(\tilde{\eta}) \geq \hat{\lambda}(\eta)$, thus $\hat{\lambda}(\tilde{\eta}) > 1/2$. By the above, this implies that $\|\tilde{\mathbf{y}} \wedge \tilde{\boldsymbol{\eta}}\| \gg \|\tilde{\mathbf{y}}\|^{-1/2}$ for any non-zero $\tilde{\mathbf{y}} \in \mathbb{Z}^{r+1}$ with $\tilde{q}(\tilde{\mathbf{y}}) = 0$. Now, let $\mathbf{x} \in \mathbb{Z}^{n+1} \setminus \{0\}$ with $q(\mathbf{x}) = 0$. Set $\mathbf{y} = T_\mathbb{R}^{-1}(\mathbf{x})$ and write $\mathbf{y} = (y_0, \ldots, y_n)$. Since $\xi \in Z^{\mathrm{li}}$, we have $\boldsymbol{\xi} \notin \ker(q)$. So, if $\|\mathbf{x} \wedge \boldsymbol{\xi}\|$ is small enough, then $\mathbf{x} \notin \ker(q)$, thus the point $\tilde{\mathbf{y}} = (y_0, \ldots, y_r)$ is non-zero with $\tilde{q}(\tilde{\mathbf{y}}) = 0$. Using

$$\|\tilde{\mathbf{y}}\| \leq \|\mathbf{y}\| \asymp \|\mathbf{x}\| \quad \text{and} \quad \|\tilde{\mathbf{y}} \wedge \tilde{\boldsymbol{\eta}}\| \leq \|\mathbf{y} \wedge \boldsymbol{\eta}\| \asymp \|\mathbf{x} \wedge \boldsymbol{\xi}\|,$$

we conclude that $\|\mathbf{x} \wedge \boldsymbol{\xi}\| \gg \|\mathbf{x}\|^{-1/2}$. $\square$

Corollary 9.4. *Let* $E = \mathbb{P}^n(\mathbb{Q}) \setminus Z(\mathbb{Q})$. *With the hypotheses of the previous proposition, we have* $\mathcal{D}_\xi(X) = \mathcal{D}_\xi(X; E)$ *for any sufficiently large* X.

Proof. By Proposition 9.3, we have $\mathcal{D}_\xi(X; Z(\mathbb{Q})) \gg X^{-1/2}$ for each $X \geq 1$. Since $\hat{\lambda}(\xi) > 1/2$, this yields $\mathcal{D}_\xi(X; Z(\mathbb{Q})) > \mathcal{D}_\xi(X)$ and so $\mathcal{D}_\xi(X) = \mathcal{D}_\xi(X; E)$, for each large enough X. $\square$

10. Quadratic forms of higher Witt index

In this last section, we assume that q is a quadratic form on $\mathbb{Q}^{n+1}$ of Witt index $m \geq 2$. Then the rank of q is at least $2m \geq 4$ and so we have $n \geq 3$. We denote by $K = \ker(q)$ the kernel of q, by $Z = Z(q_\mathbb{R})$ the associated quadratic hypersurface of $\mathbb{P}^n(\mathbb{R})$, and by $Z(\mathbb{Q})$ the set of rational points of Z. We start by proving the first assertion of Theorem 2.2 (v).

Proposition 10.1. *Suppose that a point* $\xi \in Z^{\mathrm{li}}$ *has* $\hat{\lambda}(\xi) > 1/\rho_n$. *Then for each sufficiently large* X *we have* $\mathcal{D}_\xi(X) = \mathcal{D}_\xi(X; Z(\mathbb{Q}))$.

Proof. Set $E = \mathbb{P}^n(\mathbb{Q}) \setminus Z$, and choose $\lambda \in \mathbb{R}$ with $1/\rho_n < \lambda < \hat{\lambda}(\xi)$. Then we have $\mathcal{D}_\xi(X) \leq X^{-\lambda}$ for all sufficiently large X while Theorem 2.2 (i) gives $\mathcal{D}_\xi(X; E) > X^{-\lambda}$ for arbitrarily large values of X. Altogether, this means that $\mathcal{D}_\xi(X) = \mathcal{D}_\xi(X; Z(\mathbb{Q}))$ for arbitrarily large values of X. Now, suppose that $\mathcal{D}_\xi(Y) = \mathcal{D}_\xi(Y; Z(\mathbb{Q}))$ for some $Y \geq 2$. To conclude it suffices to show that, if Y is large enough, we also have $\mathcal{D}_\xi(X) = \mathcal{D}_\xi(X; Z(\mathbb{Q}))$ for each $X \in [Y/2, Y]$.

To prove this claim, choose a zero $\mathbf{y}$ of q in $\mathbb{Z}^{n+1} \setminus \{0\}$ with $\|\mathbf{y}\| \leq Y$ and $D_\xi(\mathbf{y}) = \mathcal{D}_\xi(Y; Z(\mathbb{Q}))$. Then $\mathbf{y}$ is a primitive point of $\mathbb{Z}^{n+1}$. Choose also $X \in [Y/2, Y]$ and a point $\mathbf{x} \in \mathbb{Z}^{n+1} \setminus \{0\}$ with $\|\mathbf{x}\| \leq X$ and $D_\xi(\mathbf{x}) = \mathcal{D}_\xi(X)$. If $\|\mathbf{y}\| \leq Y/2$, we may take $\mathbf{x} = \mathbf{y}$ and thus $\mathcal{D}_\xi(X) = \mathcal{D}_\xi(X; Z(\mathbb{Q}))$. So, we may assume that $\|\mathbf{y}\| \geq Y/2$. If $\langle \mathbf{x}, \mathbf{y} \rangle_\mathbb{Q}$ has dimension 2 and is not totally isotropic, then using Corollary 9.2 we obtain

$$\frac{Y}{2} \leq \|\mathbf{y}\| \ll \|\mathbf{x} \wedge \mathbf{y}\|^2 \ll \left(\|\mathbf{x}\| D_\xi(\mathbf{y}) + \|\mathbf{y}\| D_\xi(\mathbf{x})\right)^2 \ll Y^{2(1-\lambda)}$$

and so Y is bounded from above because $\lambda > 1/\rho_n > 1/2$. Otherwise, we have $q(\mathbf{x}) = 0$ and so $\mathcal{D}_\xi(X) = \mathcal{D}_\xi(X; Z(\mathbb{Q}))$. $\square$

Under the hypotheses of the proposition, it can also be shown by a simple adaptation of the proof that, if $(\mathbf{x}_i)_{i \geq 1}$ is a sequence of minimal points for ξ with respect to $\mathbb{Z}^{n+1}$ (as defined for example in [10, §2]), then $\langle \mathbf{x}_i, \mathbf{x}_{i-1} \rangle_\mathbb{Q}$ is a totally isotropic subspace of dimension 2 for each sufficiently large i. Moreover, since $\xi \in Z^{\mathrm{li}}$, we also have $\mathbf{x}_i \notin K$ for all large enough i. We mention this to motivate the construction that we undertake below to prove the second assertion of Theorem 2.2 (v). However, before we proceed, we need the following crucial fact (a separate argument will be required for the case of rank 4).

Lemma 10.2. *Suppose that q has rank at least 5. Let $\mathbf{x} \in \mathbb{Q}^{n+1}$ be a zero of q outside of K and let $V = \langle \mathbf{x} \rangle_{\mathbb{Q}}^{\perp}$. Then, for each finite set of proper subspaces of $\mathbb{Q}^{n+1}$ not containing V, there is a zero of q in V which lies outside of each of these subspaces.*

Proof. Since $\mathbf{x}$ is a zero of q not in K, it is contained in a hyperbolic plane H_1. Since q has Witt index at least 2 and rank $r \geq 5$, we may write $\mathbb{Q}^{n+1} = H_1 \perp H_2 \perp W \perp K$ where H_2 is another hyperbolic plane and W is a non-degenerate subspace of dimension $r - 4 \geq 1$. Then, V contains the non-degenerate subspace $H_2 \perp W$ of dimension $r - 2 \geq 3$. So the restriction of q to V has rank at least 3 and Witt index at least 1. The conclusion follows by applying Proposition 4.4. $\qquad\square$

Lemma 10.3. *Suppose that a finite non-empty sequence $(\mathbf{x}_1, \ldots, \mathbf{x}_k)$ of zeros of q (in $\mathbb{Q}^{n+1}$) satisfies the following conditions for $i = 1, \ldots, k$:*

  (i) *$\mathbf{x}_i \notin K$;*
  (ii) *$\langle \mathbf{x}_i, \mathbf{x}_{i-1} \rangle_{\mathbb{Q}}$ is totally isotropic if $i \geq 2$;*
  (iii) *$\mathbf{x}_i \notin \langle \mathbf{x}_j, \ldots, \mathbf{x}_{i-1} \rangle_{\mathbb{Q}}$ where $j = \max\{1, i - n\}$ if $i \geq 2$;*
  (iv) *$\mathbf{x}_i \notin \langle \mathbf{x}_{i-(n-1)}, \ldots, \mathbf{x}_{i-1} \rangle_{\mathbb{Q}}^{\perp}$ if $i \geq n$.*

Then there exists a zero $\mathbf{x}_{k+1} \in \mathbb{Q}^{n+1}$ of q which satisfies the same conditions for $i = k + 1$. Moreover, if $\mathbf{x}_{k+1}$ is such a point, then $a(\mathbf{x}_{k+1} + b\mathbf{x}_k)$ also satisfies these conditions for any $a \in \mathbb{Q}^{\times}$ and any $b \in \mathbb{Q}$ except possibly for one value of b.

In this statement, the properties that matter for us are (i), (ii) which we motivated above, and the technical condition (iii) which implies that any $n + 1$ consecutive points among $(\mathbf{x}_1, \ldots, \mathbf{x}_k)$ are linearly independent over $\mathbb{Q}$. However, it can be checked that, when $k \geq n$, the existence of a point $\mathbf{x}_{k+1} \in \mathbb{Q}^{n+1}$ satisfying (i), (ii) and (iii) for $i = k + 1$ requires that (iv) holds for $i = k$. So, we need (iv) as well for the recurrence step.

Proof. Set $V = \langle \mathbf{x}_k \rangle_{\mathbb{Q}}^{\perp}$ and $W_1 = \langle \mathbf{x}_j, \ldots, \mathbf{x}_k \rangle_{\mathbb{Q}}$ where $j = \max\{1, k + 1 - n\}$. Set also $W_2 = \langle \mathbf{x}_{k+2-n}, \ldots, \mathbf{x}_k \rangle_{\mathbb{Q}}$ if $k \geq n - 1$. We need to show that there exists a zero of q in $V \setminus (K \cup W_1)$ if $k \leq n - 2$, in $V \setminus (K \cup W_1 \cup W_2^{\perp})$ if $k \geq n - 1$.

We first note that $\dim_{\mathbb{Q}}(V) = n$ since $\mathbf{x}_k \notin K$, that $\dim_{\mathbb{Q}}(W_1) = k - j + 1$ by Condition (iii), and that $\dim_{\mathbb{Q}}(W_2) = n - 1$ if $k \geq n - 1$ by the same condition. We deduce that $V \not\subseteq W_1$ because otherwise we would have $k \geq n$ and $V = W_1$ (by comparing dimensions), thus $\mathbf{x}_k \in W_1^{\perp} \subseteq \langle \mathbf{x}_{k+1-n}, \ldots, \mathbf{x}_{k-1} \rangle_{\mathbb{Q}}^{\perp}$, which contradicts Condition (iv) for $i = k$. If $k \geq n - 1$, we also have $V \not\subseteq W_2^{\perp}$ because otherwise we would get $W_2 \subseteq V^{\perp} = K + \mathbb{Q}\mathbf{x}_k$ which is impossible since $\dim_{\mathbb{Q}}(K) \leq n - 3$. For the same reason, we also have $V \not\subseteq K$. We conclude that $V \not\subseteq (K \cup W_1)$ if $k \leq n - 2$, and $V \not\subseteq (K \cup W_1 \cup W_2^{\perp})$ if $k \geq n - 1$.

By the above, Lemma 10.2 yields a zero $\mathbf{x}_{k+1}$ of q with the requested properties if the rank of q is at least 5. Suppose now that q has rank 4. Then its Witt index is $m = 2$.

Since $\mathbf{x}_k$ does not belong to K, it belongs to an hyperbolic plane $H = \langle \mathbf{x}_k, \mathbf{y} \rangle_{\mathbb{Q}}$ for some zero $\mathbf{y}$ of q with $b(\mathbf{x}_k, \mathbf{y}) \neq 0$. Write $\mathbb{Q}^{n+1} = K \perp H \perp H'$ where $H' = \langle \mathbf{x}', \mathbf{y}' \rangle_{\mathbb{Q}}$ is another hyperbolic plane generated by zeros $\mathbf{x}', \mathbf{y}'$ of q with $b(\mathbf{x}', \mathbf{y}') \neq 0$. This decomposition yields $V = K' \perp H'$ where $K' = K + \mathbb{Q}\mathbf{x}_k$ is the kernel of the restriction $q|_V$ of q to V. So, $q|_V$ has rank 2 and Witt index 1. Thus V admits exactly two maximal totally isotropic subspaces $U_1 = K' + \mathbb{Q}\mathbf{x}'$ and $U_2 = K' + \mathbb{Q}\mathbf{y}'$, of dimension $n - 1$, and the set of zeros of q in V is $U_1 \cup U_2$. So, we need to show that $U_1 \cup U_2$ is not contained in $K \cup W_1$ if $k \leq n - 2$, and not contained in $K \cup W_1 \cup W_2^{\perp}$ if $k \geq n - 1$. For $k \leq n - 2$, this is clear since K and W_1 have dimensions strictly smaller than $n - 1$. Thus we may assume that $k \geq n - 1$. Since $U_1 + U_2 = V$ is not contained in W_1 nor in $W_2^{\perp}$, there is at most one index $r \in \{1, 2\}$ such that $U_r \subseteq W_1$ and at most one index $s \in \{1, 2\}$ such that $U_s \subseteq W_2^{\perp}$. If such r and s exist and are distinct, we may assume that $r = 1$ and $s = 2$ by permuting $\mathbf{x}'$ and $\mathbf{y}'$ is necessary. Then, we have $U_1 \subseteq W_1$ and $W_2 \subseteq U_2^{\perp} = U_2$, so W_2 and U_2 coincide since they have the same dimension $n - 1$. This is impossible because W_1 contains W_2 but not U_2. Thus either r or s does not exist or they are equal. This means that, for some $t \in \{1, 2\}$, we have both $U_t \nsubseteq W_1$ and $U_t \nsubseteq W_2^{\perp}$, so $U_t \nsubseteq K \cup W_1 \cup W_2^{\perp}$ (since $\dim_{\mathbb{Q}}(K) < n - 1$), and we are done.

Finally, suppose that a zero $\mathbf{x}_{k+1}$ of q satisfies Conditions (i) to (iv) for $i = k + 1$. Then the point $a(\mathbf{x}_{k+1} + b\mathbf{x}_k)$ is a zero of q which satisfies the same conditions for all $a \in \mathbb{Q}^{\times}$ and all $b \in \mathbb{Q}$ such that $\mathbf{x}_{k+1} + b\mathbf{x}_k$ does not belong to K if $k \leq n - 2$ or to $W_2^{\perp}$ if $k \geq n - 1$. Since in both cases the point $\mathbf{x}_{k+1}$ does not lie in that subspace, this excludes at most one value of b. $\qquad\square$

We conclude with the proof of the following assertion from Theorem 2.2 (v).

Proposition 10.4. *Let $\varphi \colon [1, \infty) \to (0, 1]$ be a monotonically decreasing function with*

$$\lim_{X \to \infty} \varphi(X) = 0 \quad \text{and} \quad \lim_{X \to \infty} X\varphi(X) = \infty.$$

Then there exist uncountably many points $\xi \in Z^{\mathrm{li}}$ which satisfy $\mathcal{D}_{\xi}(X; Z(\mathbb{Q})) \leq \varphi(X)$ for all sufficiently large X.

Proof. Starting with a zero $\mathbf{x}_1$ of q in $\mathbb{Z}^{n+1} \setminus K$, Lemma 10.3 allows us to construct recursively a sequence $(\mathbf{x}_i)_{i \geq 1}$ of zeros of q in $\mathbb{Z}^{n+1}$ which satisfy Conditions (i) to (iv) of that lemma for each $i \geq 1$, such that, upon setting $X_i = \|\mathbf{x}_i\|$ for each $i \geq 1$, we also have

(v) $X_i > X_{i-1}$ when $i \geq 2$;
(vi) $\mathrm{dist}(\mathbf{x}_i, \mathbf{x}_{i-1}) \leq \dfrac{1}{3} \min \left\{ 2X_{i-1}^{-1}\varphi(X_i), \mathrm{dist}(\mathbf{x}_{i-1}, \mathbf{x}_{i-2}) \right\}$ when $i \geq 3$.

Indeed, suppose that $\mathbf{x}_1, \ldots, \mathbf{x}_{i-1}$ have been constructed for some $i \geq 2$. Then Lemma 10.3 provides a zero $\mathbf{x}_i$ of q satisfying Conditions (i) to (iv) of that lemma. Upon multiplying it by a suitable positive integer, we may assume that $\mathbf{x}_i \in \mathbb{Z}^{n+1}$. Let $\tilde{\mathbf{x}}_i$ denote this particular

zero. By Lemma 10.3, the point $\mathbf{x}_i = \tilde{\mathbf{x}}_i + b\mathbf{x}_{i-1}$ also satisfies these conditions for all but at most one value of b. We find

$$\operatorname{dist}(\mathbf{x}_i, \mathbf{x}_{i-1}) = \frac{\|\tilde{\mathbf{x}}_i \wedge \mathbf{x}_{i-1}\|}{X_{i-1} X_i}$$

with a numerator that is independent of the choice of b. As both $X_i = \|\mathbf{x}_i\|$ and $X_i \varphi(X_i)$ go to infinity with $|b|$, Conditions (v) and (vi) are fulfilled for $|b|$ large enough.

In $\mathbb{P}^n(\mathbb{R})$, the image $([\mathbf{x}_i])_{i\geq 1}$ of such a sequence converges to a point ξ with

$$(10.1) \qquad \operatorname{dist}(\xi, [\mathbf{x}_{i-1}]) \leq \sum_{j=i}^{\infty} \operatorname{dist}(\mathbf{x}_j, \mathbf{x}_{j-1}) \leq \operatorname{dist}(\mathbf{x}_i, \mathbf{x}_{i-1}) \sum_{j=0}^{\infty} 3^{-j} = \frac{3}{2} \operatorname{dist}(\mathbf{x}_i, \mathbf{x}_{i-1})$$

for each $i \geq 2$. Since $[\mathbf{x}_i] \in Z(\mathbb{Q}) \subset Z$ for each $i \geq 1$ and since Z is a closed subset of $\mathbb{P}^n(\mathbb{R})$, the point ξ belongs to Z. When $i \geq 3$, Condition (vi) combined with (10.1) yields

$$(10.2) \qquad D_\xi(\mathbf{x}_{i-1}) = X_{i-1} \operatorname{dist}(\xi, [\mathbf{x}_{i-1}]) \leq \frac{3}{2} X_{i-1} \operatorname{dist}(\mathbf{x}_i, \mathbf{x}_{i-1}) \leq \varphi(X_i).$$

In particular, we have $\lim_{i\to\infty} D_\xi(\mathbf{x}_i) = 0$, and so Lemma 6.2 implies that $\xi \in Z^{\mathrm{li}}$ because any $n+1$ consecutive points $\mathbf{x}_i, \ldots, \mathbf{x}_{i+n}$ form a basis of $\mathbb{Q}^{n+1}$. Finally, for any $X \geq X_2$, there exists an index $i \geq 3$ such that $X_{i-1} \leq X < X_i$ and using (10.2) we obtain

$$\mathcal{D}_\xi(X; Z(\mathbb{Q})) \leq \mathcal{D}_\xi(X_{i-1}; Z(\mathbb{Q})) \leq D_\xi(\mathbf{x}_{i-1}) \leq \varphi(X_i) \leq \varphi(X).$$

Thus, any sequence $(\mathbf{x}_i)_{i\geq 1}$ as above yields a point $\xi \in Z^{\mathrm{li}}$ with the requested property. To show that there are uncountably many such points, consider any sequence $\xi_1, \xi_2, \ldots$ of these. Then choose $(\mathbf{x}_i)_{i\geq 1}$ in $\mathbb{Z}^{n+1}$ satisfying Conditions (i) to (vi) as well as

$$(10.3) \qquad \operatorname{dist}(\mathbf{x}_i, \mathbf{x}_{i-1}) < \frac{2}{3} \min_{1\leq j\leq i} \operatorname{dist}(\xi_j, [\mathbf{x}_{i-1}])$$

for each $i \geq 2$. This is possible since none of the points ξ_j is rational and therefore the right hand side of the last inequality is non-zero. Let $\xi = \lim_{i\to\infty} [\mathbf{x}_i] \in \mathbb{P}^n(\mathbb{R})$. Combining (10.1) and (10.3) for an arbitrary $i \geq 3$, we obtain

$$\operatorname{dist}(\xi, [\mathbf{x}_{i-1}]) < \min_{1\leq j\leq i} \operatorname{dist}(\xi_j, [\mathbf{x}_{i-1}]),$$

thus $\xi \notin \{\xi_1, \ldots, \xi_i\}$. So ξ does not belong to the sequence and therefore the set of points constructed in this way is uncountable. $\qquad\square$

Appendix A. An approximation lemma

The following lemma is needed in Step 2 of Section 8. It certainly occurs in the literature in various forms. By lack of an appropriate reference, we provide a short proof below.

Lemma A.1. *Let V be a real or complex inner product space of finite positive dimension, let $T\colon V \to V$ be a linear operator on V, let $(\mathbf{v}_i)_{i\geq 1}$ be a sequence in V, and let $\alpha \in \mathbb{C}$ with $|\alpha| > 1$. Suppose that any root $\beta \in \mathbb{C}$ of the minimal polynomial of T has $|\beta| < 1$ or is simple with $|\beta| = 1$ or is simple with $\beta = \alpha$. Suppose also that*

$$\sum_{i=1}^{\infty} \|\mathbf{v}_{i+1} - T(\mathbf{v}_i)\| < \infty$$

for the norm $\| \ \|$ of V. Then there exists a vector $\mathbf{v} \in V$ (possibly zero) and a constant $C > 0$ satisfying $T(\mathbf{v}) = \alpha\mathbf{v}$ and $\|\mathbf{v}_i - \alpha^i\mathbf{v}\| \leq C$ for each $i \geq 1$.

Recall that the minimal polynomial of T is the monic polynomial $m_T(x)$ of $\mathbb{C}[x]$ of smallest degree such that $m_T(T) = 0$. It is a divisor of the characteristic polynomial of T with the same set of roots in $\mathbb{C}$. If $m_T(\alpha) \neq 0$, the vector $\mathbf{v}$ produced by the lemma is zero and the conclusion means that the sequence $(\mathbf{v}_i)_{i\geq 1}$ is bounded.

Proof. Upon extending T by linearity to $\mathbb{C}\otimes_{\mathbb{R}} V$ in the case where V is real, we reduce to the situation where V is complex. As V decomposes as a direct sum of irreducible T-invariant subspaces, it suffices to prove the lemma when V itself is irreducible. Then T admits a single eigenvalue β and its minimal polynomial is $(x - \beta)^n$ for some $n > 0$. Put

$$\mathbf{u}_1 = \mathbf{v}_1 \quad \text{and} \quad \mathbf{u}_{k+1} = \mathbf{v}_{k+1} - T(\mathbf{v}_k) \quad \text{for each } k \geq 1.$$

By hypothesis, the sum $c_1 = \sum_{k\geq 1} \|\mathbf{u}_k\|$ is finite. Moreover, we have

$$(\text{A.1}) \qquad\qquad \mathbf{v}_i = \sum_{k=1}^{i} T^{i-k}(\mathbf{u}_k) \quad (i \geq 1).$$

If $|\beta| < 1$, the operator norm $\|T^k\|$ tends to 0 as $k \to \infty$. Thus there exists a constant $c_2 \geq 1$ such that $\|T^k\| \leq c_2$ for all $k \geq 0$. If $|\beta| = 1$, we have $n = 1$ thus $\|T^k\| = 1$ for each $k \geq 0$, and we may simply take $c_2 = 1$. In both cases, this yields

$$\|\mathbf{v}_i\| \leq c_2 \sum_{k=1}^{i} \|\mathbf{u}_k\| \leq c_1 c_2 \quad (i \geq 1)$$

and so the conclusion holds with $\mathbf{v} = 0$. Finally if $|\beta| > 1$, we have $\beta = \alpha$ and $n = 1$. In this case the series $\mathbf{v} = \sum_{k=1}^{\infty} \alpha^{-k}\mathbf{u}_k$ converges in V. It satisfies $T(\mathbf{v}) = \alpha\mathbf{v}$ since $T = \alpha I$ and, by (A.1), we find as requested

$$\left\|\mathbf{v}_i - \alpha^i\mathbf{v}\right\| = \left\|\sum_{k=i+1}^{\infty} \alpha^{i-k}\mathbf{u}_k\right\| \leq \sum_{k=i+1}^{\infty} \|\mathbf{u}_k\| \leq c_1 \quad (i \geq 1). \qquad \square$$

References

[1] Y. Bugeaud, M. Laurent, Exponents of Diophantine approximation and Sturmian continued fractions, *Ann. Inst. Fourier (Grenoble)* **55** (2005), 773-804.

[2] T. Das, L. Fishman, D. Simmons, and M. Urbański A variational principle in the parametric geometry of numbers, preprint, 81 pages; arXiv:1901.06602 [math.NT].

[3] H. Davenport and W. M. Schmidt, Approximation to real numbers by algebraic integers, *Acta Arith.* **15** (1969), 393–416.

[4] É. Gaudron; G. Rémond, Espaces adéliques quadratiques, *Math. Proc. Cambridge Philos. Soc.* **162** (2017), 211–247.

[5] D. Kleinbock and N. Moshchevitin, Simultaneous Diophantine approximation: sums of squares and homogeneous polynomials, *Acta Arith.* **190** (2019), 87–100.

[6] S. Lang, *Algebra*, Second edition, Addison-Wesley Publishing Co., 1965, 526 pp.

[7] M. Laurent, On transfer inequalities in Diophantine Approximation, in: Analytic Number Theory in Honour of Klaus Roth, Cambridge University Press (2009), 306–314.

[8] A. Marnat and N. Moshchevitin, An optimal bound for the ratio between ordinary and uniform exponents of Diophantine approximation, *Mathematika* **66** (2020), 818–854.

[9] M. R. Murty, Polynomials assuming square values, in: Number theory and discrete geometry, pp.155–163, Ramanujan Math. Soc. Lect. Notes Ser. **6**, Ramanujan Math. Soc., Mysore, 2008.

[10] V. Nguyen, A. Poëls and D. Roy, A transference principle for simultaneous rational approximation, *J. Théor. Nombres Bordeaux* (to appear), 13 pages; arXiv:1908.11777 [math.NT].

[11] G. Pólya and G. Szegő, Problems and theorems in analysis II: Theory of functions, zeros, polynomials, determinants, number theory, geometry. Translated from the German by C. E. Billigheimer, Springer-Verlag, Berlin, 1976, xii+392 pp.

[12] D. Roy, Approximation simultanée d'un nombre et de son carré, *C. R. Acad. Sci., Paris, ser. I* **336** (2003), 1–6.

[13] D. Roy, Approximation to real numbers by cubic algebraic integers I, *Proc. London Math. Soc.* **88** (2004), 42–62.

[14] D. Roy, Rational approximation to real points on conics, *Ann. Inst. Fourier (Grenoble)* **63** (2013), 2331–2348.

[15] D. Roy, On Schmidt and Summerer parametric geometry of numbers, *Ann. of Math.* **182** (2015), 739–786.

[16] W. M. Schmidt and L. Summerer, Diophantine approximation and parametric geometry of numbers, *Monatsh. Math.* **169** (2013), 51–104.

[17] W. M. Schmidt and L. Summerer, Simultaneous approximation to three numbers, *Mosc. J. Comb. Number Theory* **3** (2013), 84–107.

[18] J.-P. Serre, *A course in arithmetic*, Graduate Texts in Math., vol. 7, Springer-Verlag, 1973, 115 pp.

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